A Standard Theory for the Pure Lambda-Value Calculus

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Standard theory in the classical lambda calculus (a.k.a. Barendregt's theme)

- Pure untyped lambda calculus (λK) with reduction ($\rightarrow_\beta^*$) and conversion ($=_\beta$) theories.
- Solvability:

  M is solvable iff there exists $\mathbf{C}_h[\ ] \equiv (\lambda x_1 \ldots x_m[\ ])N_1 \ldots N_n$ such that $\mathbf{C}_h[M] =_\beta I$. [Barendregt, 1984]
- Unsolvable terms denote $\perp$ in D_∞ [Wadsworth, 1976].
- $\mathcal{H} = \{M =_\beta N \mid M, N \in \Lambda^0 \text{ unsolvables}\}^+$ is a lambda theory [Barendregt, 1984].
- D_∞ satifies $\mathcal{H}$, i.e., D_∞ is *sensible* [Barendregt, 1984].
- Quasi-leftmost reduction characterises complete strategies [Hindley and Seldin, 2008].

Solvability and transformability in λK

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Theorem ([Wadsworth, 1976])

Let $N \in \mathrm{NF}$. For every M, there exists $\mathbf{C}_h[\ ]$ such that $\mathbf{C}_h[N] \to_\beta^* M$.

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Theorem ([Wadsworth, 1976])

Let $N \in \mathrm{NF}$. For every M, there exists $\mathbf{C}_h[\ ]$ such that $\mathbf{C}_h[N] \to_\beta^* M$.

M is solvable iff it has a head normal form.

Definition (Needed Redex in λK [Barendregt et al., 1987])

A redex R in $M \equiv \mathbf{C}[R]$ is needed iff R (or some residual of it) is contracted in every reduction sequence ending in normal form:

$$M \equiv \mathbf{C}[R] \rightarrow_\beta \ldots \rightarrow_\beta N \in \mathsf{NF}$$

Polarity in λK

$$
\begin{array}{rcl}
\mathcal{P}(x) & = & x^{+} \\
\mathcal{P}(\lambda x.B) & = & (\lambda x.\mathcal{P}(B))^{+} \\
\mathcal{P}(M\,N) & = & (\mathcal{P}(M)\,\mathcal{N}(N))^{+}
\end{array}
$$

$$
\begin{array}{rcl}
\mathcal{N}(x) & = & x^{-} \\
\mathcal{N}(\lambda x.B) & = & (\lambda x.\mathcal{N}(B))^{-} \\
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\end{array}
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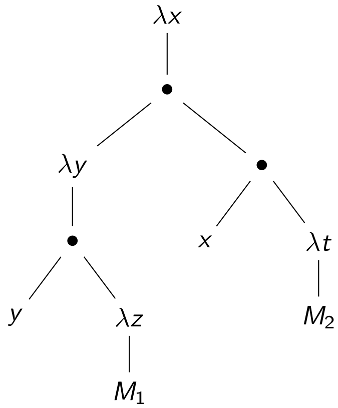
$$\lambda x.(\lambda y.y\,(\lambda z.M_1))(x\,M_2)$$



Polarity in λK

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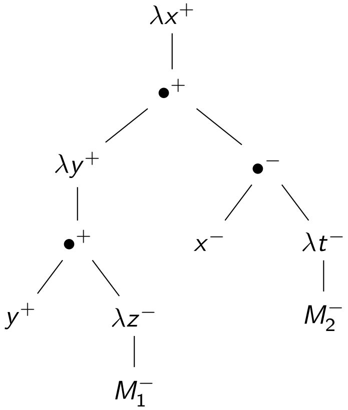
\end{aligned}$$

$$(\lambda x.((\lambda y.y^+\,(\lambda z.M_1^-)^-)^+(x^-\,M_2^-)^-)^+)^+$$



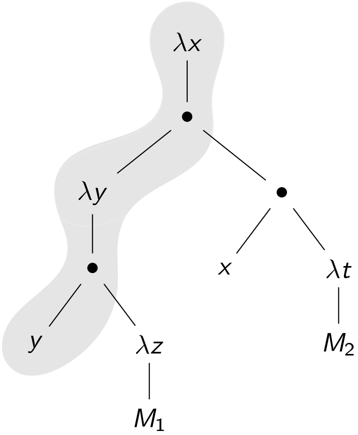
Polarity in λK (a.k.a. head spine [Barendregt et al., 1987])

$$\begin{array}{rcl} \mathcal{P}(x) & = & x^{+} \\ \mathcal{P}(\lambda x.B) & = & (\lambda x.\mathcal{P}(B))^{+} \\ \mathcal{P}(M\,N) & = & (\mathcal{P}(M)\,\mathcal{N}(N))^{+} \end{array}$$

$$\begin{array}{rcl} \mathcal{N}(x) & = & x^{-} \\ \mathcal{N}(\lambda x.B) & = & (\lambda x.\mathcal{N}(B))^{-} \\ \mathcal{N}(M\,N) & = & (\mathcal{N}(M)\,\mathcal{N}(N))^{-} \end{array}$$

$$\begin{array}{rcl} \mathrm{hs}(x) & = & \underline{x} \\ \mathrm{hs}(\lambda x.B) & = & \underline{\lambda x.}\mathrm{hs}(B) \\ \mathrm{hs}(M\,N) & = & \mathrm{hs}(M)N \end{array}$$

$\underline{\lambda x.(\lambda y.y}\,(\lambda z.M_1))(x\,M_2)$



Polarity in λK (head reduction)

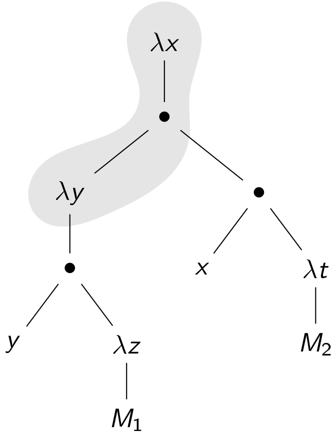
$$
\begin{aligned}
\mathcal{P}(x) &= x^+ \\
\mathcal{P}(\lambda x.B) &= (\lambda x.\mathcal{P}(B))^+ \\
\mathcal{P}(M\,N) &= (\mathcal{P}(M)\,\mathcal{N}(N))^+
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{N}(x) &= x^- \\
\mathcal{N}(\lambda x.B) &= (\lambda x.\mathcal{N}(B))^- \\
\mathcal{N}(M\,N) &= (\mathcal{N}(M)\,\mathcal{N}(N))^-
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{hr}(x) &= \underline{x} \\
\mathrm{hr}(\lambda x.B) &= \underline{\lambda x.}\mathrm{hr}(B) \\
\mathrm{hr}(M\,N) &= \mathrm{bn}(M)N
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{bn}(x) &= \underline{x} \\
\mathrm{bn}(\lambda x.B) &= \lambda x.B \\
\mathrm{bn}(M\,N) &= \mathrm{bn}(M)N
\end{aligned}
$$

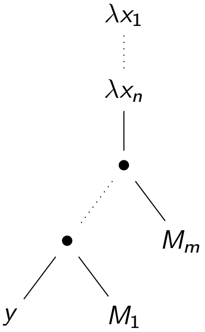
$\underline{\lambda x.(\lambda y.}y\,(\lambda z.M_1))(x\,M_2)$



Head normal forms

$$
\begin{array}{lcl}
\text{HNF} & ::= & x \\
 & | & \lambda x.\text{HNF} \\
 & | & \text{HNF}\,\Lambda
\end{array}
$$

$$\lambda x_1 \dots x_n . y\, M_1 \dots M_m$$



Pure lambda-value calculus (λ_V) [Egidi et al., 1991]

- λ_V in [Plotkin, 1975] without primitive constants and δ-rules.

$$\frac{N \in \mathsf{Val}}{(\lambda x.B)N =_{\beta_{\mathbf{v}}} [N/x]B}\ (\beta_{\mathbf{v}}) \qquad \mathsf{Val} \ ::= \ x \mid \lambda x.\Lambda$$

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- Why values as non applications?

  An irreducible application may become a divergent term when the free variables in it are substituted by other terms [Plotkin, 1975].

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- Why values as non applications?

  An irreducible application may become a divergent term when the free variables in it are substituted by other terms [Plotkin, 1975].

  "Preserving confluence by preserving potential divergence".

Is there a standard theory for λ_V?

- Call-by-value solvability [Paolini and Ronchi Della Rocca, 1999]:
  M is v-solvable iff there exists $\mathbf{C}_h[\,] \equiv (\lambda x_1 \dots x_m[\,])N_1 \dots N_n$ such that $\mathbf{C}_h[M] =_{\beta_\mathbf{v}} I$.
  The order n of a v-unsolvable M informs about the number of preceding lambdas in the term, i.e., $M =_{\beta_\mathbf{v}} \lambda x_1 \dots \lambda x_n.B$.
- The v-unsolvables of order 0 can be equated in a consistent way [Paolini and Ronchi Della Rocca, 1999].
- Domain H constructed from canonical solution of equation $D \cong [D \to_\perp D]_\perp$ [Egidi et al., 1991].
  The v-unsolvable terms of order 0 denote $\perp$ in H.
- Standard reduction in λ_V [Plotkin, 1975] is complete.
  Does 'complete' imply 'standard'?

Objection on λ_V normal forms in [Paolini and Ronchi Della Rocca, 1999]

Ω and $U \equiv (\lambda y.\Delta)(x\, I)\Delta$ are observationally equivalent.

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Consider the context $(\lambda x.[\ ])V$

$(\lambda x.\Omega)V$

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$$(\lambda x.\Omega)V \to_{\beta_{\boldsymbol{V}}} \Omega$$

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$(\lambda x.(\lambda y.\Delta)(x\, I)\Delta)V$

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$$\to_{\beta_{\mathbf{v}}} \Delta\,\Delta \equiv \Omega \to_{\beta_{\mathbf{v}}} \dots$$

But. . .

- Differences in *sequentiality*.

Solvability and transformability in λ_V

M is solvable iff there exists $\mathbf{C}_h[\ ] \equiv (\lambda x_1 \dots x_m[\ ])N_1 \dots N_n$ such that $\mathbf{C}_h[M] =_{\beta_{\boldsymbol{V}}} I$.

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Theorem (Transformability for λ_V)

$\ldots$

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The $U \equiv (\lambda y.\Delta)(x\, I)\Delta$ is a counterexample which is not transformable.

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Theorem (Transformability for λ_V)

$\ldots$

The $U \equiv (\lambda y.\Delta)(x\, I)\Delta$ is a counterexample which is not transformable.

Our research

- Reconsider definition of solvability in λ_V:
  M is solvable iff there exists $\mathbf{C}_h[\ ] \equiv (\lambda x_1 \dots x_m[\ ])N_1 \dots N_n$ such that $\mathbf{C}_h[M] =_{\beta_{\mathbf{v}}} N \in \mathrm{NF}_V$.

  $$\text{solvable} = \text{transformable} + \text{freezable}$$

  A term which is freezable into a NF_V is also operational relevant!
- Generalisation of *needed reduction* [Barendregt et al., 1987] to λ_V.
- $\mathcal{H}_V = \{M =_{\beta_{\mathbf{v}}} N \mid M, N \in \Lambda^0 \text{ unsolvables of the same order}\}^{+\mathbf{v}}$ is a consistent theory.

  E.g., $\mathcal{H}_V \nvdash \lambda x.\Omega =_{\beta_{\mathbf{v}}} \lambda x.(\lambda y.\Delta)(x\, I)\Delta$.
- A theory or a model is ω*-sensible* if it satisfies $\mathcal{H}_V$.
- *Hereditary quasi-chest reduction* (see next slides) characterises complete strategies.

  Now 'complete' implies 'hereditary quasi-chest'!

Definition (Needed Redex in λ_V [Our research])

$M \equiv \mathbf{C}[R]$ is needed iff R (or some residual of it) is contracted in every reduction sequence ending in λ_V normal form:

$$M \equiv \mathbf{C}[R] \rightarrow_{\beta_{\mathbf{V}}} \ldots \rightarrow_{\beta_{\mathbf{V}}} N \in \mathsf{NF}_V$$

Polarity in λ_V

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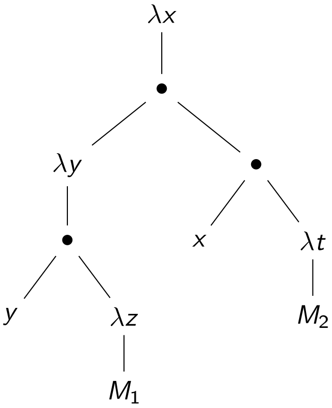
\mathcal{P}(M\,N) & = & (\mathcal{P}(M)\,\mathcal{N}(N))^+ \\
 & & \\
\mathcal{N}(x) & = & x^- \\
\mathcal{N}(\lambda x.B) & = & (\lambda x.\mathcal{N}(B))^- \\
\mathcal{N}(M\,N) & = & (\mathcal{N}(M)\,\mathcal{N}(N))^-
\end{array}
$$

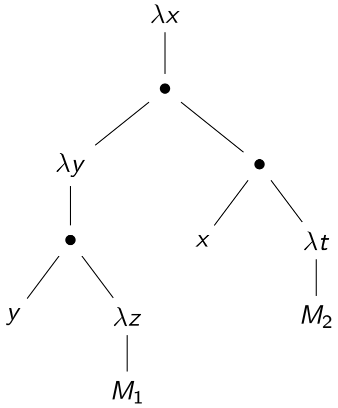
$$\lambda x.(\lambda y.y(\lambda z.M_1))(x(\lambda t.M_2))$$



Polarity in λ_V

$$
\begin{array}{rcl}
\mathcal{P}(x) & = & x^+ \\
\mathcal{P}(\lambda x.B) & = & (\lambda x.\mathcal{P}(B))^+ \\
\mathcal{P}(M\,N) & = & (\mathcal{P}(M)\,\mathcal{A}(N))^+ \\
\\
\mathcal{A}(x) & = & x^- \\
\mathcal{A}(\lambda x.B) & = & (\lambda x.\mathcal{B}(B))^- \\
\mathcal{A}(M\,N) & = & (\mathcal{A}(M)\,\mathcal{A}(N))^- \\
\\
\mathcal{B}(x) & = & x^- \\
\mathcal{B}(\lambda x.B) & = & (\lambda x.\mathcal{B}(B))^- \\
\mathcal{B}(M\,N) & = & (\mathcal{B}(M)\,\mathcal{B}(N))^-
\end{array}
$$

$$\lambda x.(\lambda y.y(\lambda z.M_1))(x(\lambda t.M_2))$$



Polarity in λ_V

$$
\begin{aligned}
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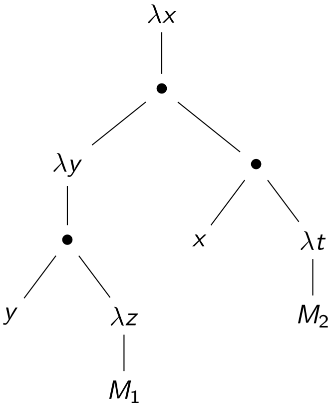
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}(x) &= x^- \\
\mathcal{A}(\lambda x.B) &= (\lambda x.\mathcal{B}(B))^- \\
\mathcal{A}(M\,N) &= (\mathcal{A}(M)\,\mathcal{A}(N))^-
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{B}(x) &= x^- \\
\mathcal{B}(\lambda x.B) &= (\lambda x.\mathcal{B}(B))^- \\
\mathcal{B}(M\,N) &= (\mathcal{B}(M)\,\mathcal{B}(N))^-
\end{aligned}
$$

$$\lambda x.(\lambda y.y(\lambda z.M_1))(x(\lambda t.M_2))$$



Polarity in λ_V

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\begin{aligned}
\mathcal{P}(x) &= x^+ \\
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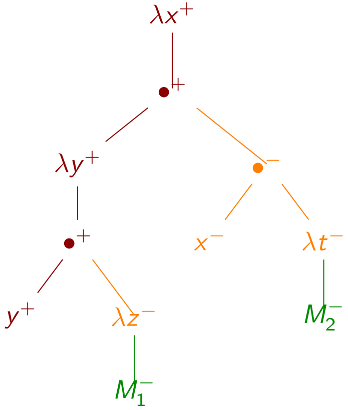
\mathcal{B}(M\,N) &= (\mathcal{B}(M)\,\mathcal{B}(N))^-
\end{aligned}
$$

$$(\lambda x.(\lambda y.y^+(\lambda z.M_1^-)^-)^+(x^-(\lambda t.M_2^-)^-)^-)^+$$



Polarity in λ_V (ribcage reduction)

$$
\begin{array}{rcl}
\mathcal{P}(x) & = & x^+ \\
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\\
\mathcal{A}(x) & = & x^- \\
\mathcal{A}(\lambda x.B) & = & (\lambda x.\mathcal{B}(B))^- \\
\mathcal{A}(M\,N) & = & (\mathcal{A}(M)\,\mathcal{A}(N))^- \\
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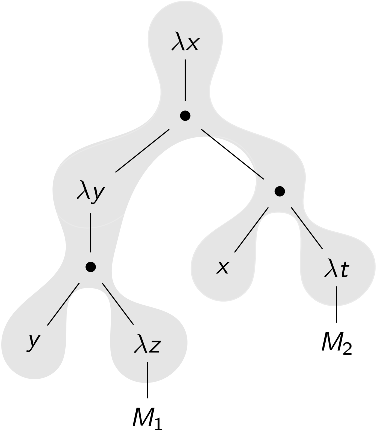
\mathcal{B}(x) & = & x^- \\
\mathcal{B}(\lambda x.B) & = & (\lambda x.\mathcal{B}(B))^- \\
\mathcal{B}(M\,N) & = & (\mathcal{B}(M)\,\mathcal{B}(N))^-
\end{array}
$$

$$
\begin{array}{rcl}
\mathsf{rc}(x) & = & \underline{x} \\
\mathsf{rc}(\lambda x.B) & = & \underline{\lambda x}.\mathsf{rc}(B) \\
\mathsf{rc}(M\,N) & = & \mathsf{rc}(M)\mathsf{bv}(N) \\
\\
\mathsf{bv}(x) & = & \underline{x} \\
\mathsf{bv}(\lambda x.B) & = & \underline{\lambda x}.B \\
\mathsf{bv}(M\,N) & = & \mathsf{bv}(M)\mathsf{bv}(N)
\end{array}
$$

$\underline{\lambda x.(\lambda y.y(\lambda z.}M_1))\underline{(x(\lambda t.}M_2))$



Polarity in λ_V (chest reduction)

$$\begin{aligned}
\mathcal{P}(x) &= x^{+} \\
\mathcal{P}(\lambda x.B) &= (\lambda x.\mathcal{P}(B))^{+} \\
\mathcal{P}(M\,N) &= (\mathcal{P}(M)\,\mathcal{A}(N))^{+}
\end{aligned}$$

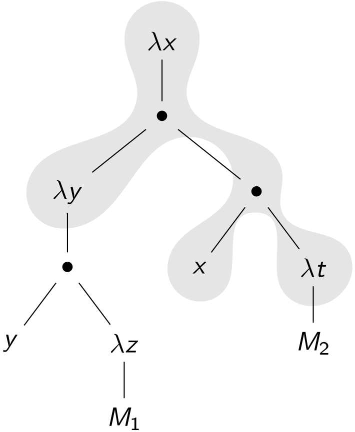
$$\begin{aligned}
\mathcal{A}(x) &= x^{-} \\
\mathcal{A}(\lambda x.B) &= (\lambda x.\mathcal{B}(B))^{-} \\
\mathcal{A}(M\,N) &= (\mathcal{A}(M)\,\mathcal{A}(N))^{-}
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}(x) &= x^{-} \\
\mathcal{B}(\lambda x.B) &= (\lambda x.\mathcal{B}(B))^{-} \\
\mathcal{B}(M\,N) &= (\mathcal{B}(M)\,\mathcal{B}(N))^{-}
\end{aligned}$$

$$\begin{aligned}
\mathrm{ch}(x) &= \underline{x} \\
\mathrm{ch}(\lambda x.B) &= \underline{\lambda x.}\mathrm{ch}(B) \\
\mathrm{ch}(M\,N) &= \mathrm{bv}(M)\mathrm{bv}(N)
\end{aligned}$$

$$\begin{aligned}
\mathrm{bv}(x) &= \underline{x} \\
\mathrm{bv}(\lambda x.B) &= \underline{\lambda x.}B \\
\mathrm{bv}(M\,N) &= \mathrm{bv}(M)\mathrm{bv}(N)
\end{aligned}$$

$$\underline{\lambda x.(\lambda y.}y(\lambda z.M_1))\underline{(x(\lambda t.}M_2))$$



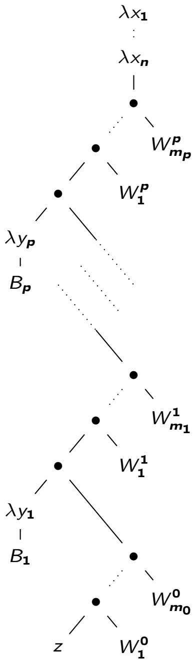
Chest normal forms

$$
\begin{array}{rcl}
\mathsf{CHNF} & ::= & x \\
& | & \lambda x.\mathsf{CHNF} \\
& | & \mathsf{Stuck} \\
\mathsf{WNF_V} & ::= & \mathsf{Val} \\
& | & \mathsf{Stuck}
\end{array}
\qquad
\begin{array}{rcl}
\mathsf{Val} & ::= & x \\
& | & \lambda x.\Lambda \\
\mathsf{Stuck} & ::= & x\,\mathsf{WNF_V} \\
& | & (\lambda x.\Lambda)\mathsf{Stuck} \\
& | & \mathsf{Stuck}\,\mathsf{WNF_V}
\end{array}
$$

Chest normal forms

$$
\begin{array}{rcl}
\text{CHNF} & ::= & x \\
 & | & \lambda x.\text{CHNF} \\
 & | & \text{Stuck} \\
\text{WNF}_\text{V} & ::= & \text{Val} \\
 & | & \text{Stuck}
\end{array}
\qquad
\begin{array}{rcl}
\text{Val} & ::= & x \\
 & | & \lambda x.\Lambda \\
\text{Stuck} & ::= & x\,\text{WNF}_\text{V} \\
 & | & (\lambda x.\Lambda)\text{Stuck} \\
 & | & \text{Stuck}\,\text{WNF}_\text{V}
\end{array}
$$

$$
\begin{array}{l}
\lambda x_1 \ldots x_n. \\
\quad (\lambda y_p.B_p) \\
\qquad (\ \ldots \\
\qquad\quad ((\lambda y_1.B_1)(z\, W_1^0 \ldots W_{m_0}^0)\, W_1^1 \ldots W_{m_1}^1) \\
\qquad \ldots\ )\, W_1^p \ldots M_{m_p}^p
\end{array}
$$



Semi-decision procedure for λ_V solvability

$$(\lambda x. x\,\Delta)(x\,I)\Delta\,(\lambda x.\Omega)$$

Semi-decision procedure for λ_V solvability

$$(\lambda x.x\,\Delta)(x\,I)\Delta\,(\lambda x.\Omega)$$

- Mark-test-contract algorithm
  - Positive subterms checked for solvability, negative subterms checked only for *valuability*.
  - Stops anytime a (sub)term is freezable, avoiding to diverge when it is not transformable.
- On-line implementation at `http://babel.ls.fi.upm.es/~agarcia/talks/DomainsXI`

To summarise

- Reconsider the definition of solvability in λ_V.
- The λ_v normal forms informs about sequentiality.
- Generalisation of needed reduction to λ_V.
- Hereditary chest reduction characterises complete strategies in λ_V.
- Semi-decision procedure for solvability in λ_V.
- Consistent theory $\mathcal{H}_V$ equates unsolvables of the same order.
- A theory or model is ω-sensible iff it satisfies $\mathcal{H}_V$.

Full abstraction

Questions:

- Fully abstract model w.r.t. $\mathcal{H}_V$?
- Böhm trees for λ_V?

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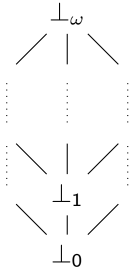
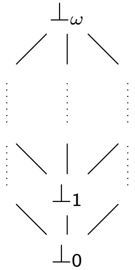


Full abstraction

Questions:

- Fully abstract model w.r.t. $\mathcal{H}_V$?
- Böhm trees for λ_V?

Thanks!



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Backup

$$\underline{(\lambda x.(\lambda y.z)(x\,\Delta))\Delta} \qquad \text{(recall } \Delta \equiv (\lambda x.x\,x))$$

$$(\lambda x.(\lambda y.z)(x\,\Delta))\Delta \qquad (\text{recall } \Delta \equiv (\lambda x.x\,x))$$

$$(\lambda x.(\lambda y.z)(x\,\Delta))\Delta \qquad \text{(recall } \Delta \equiv (\lambda x.x\,x))$$

$$(\lambda x.\underline{(\lambda y.z)(x\,\Delta)})\Delta \qquad \text{(recall } \Delta \equiv (\lambda x.x\,x))$$

$$(\lambda x.\underline{(\lambda y.z)(x\,\Delta)})\Delta \qquad \text{(recall } \Delta \equiv (\lambda x.x\,x))$$

$$\swarrow$$

$$(\lambda x.z)\Delta$$

$$(\lambda x.\underline{(\lambda y.z)(x\,\Delta)})\Delta \qquad \text{(recall } \Delta \equiv (\lambda x.x\,x))$$

$$\swarrow$$

$$\underline{(\lambda x.z)\Delta}$$

$$(\lambda x.\underline{(\lambda y.z)(x\,\Delta)})\Delta \qquad \text{(recall } \Delta \equiv (\lambda x.x\,x))$$

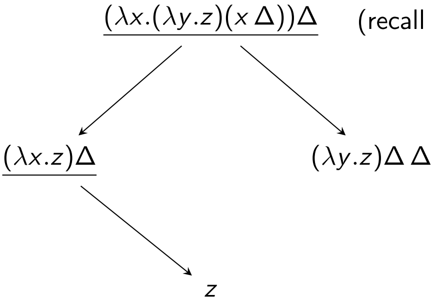
$$\longrightarrow \underline{(\lambda x.z)\Delta}$$

$$\longrightarrow z$$

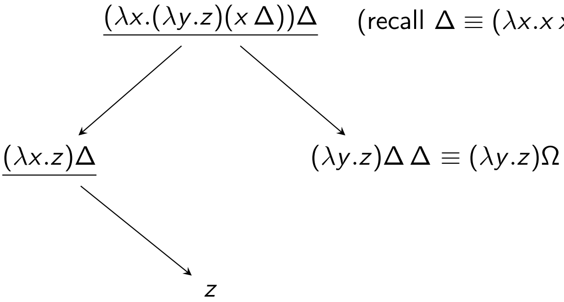
$\underline{(\lambda x.(\lambda y.z)(x\,\Delta))\Delta}$ (recall $\Delta \equiv (\lambda x.x\,x)$)

$\underline{(\lambda x.z)\Delta}$

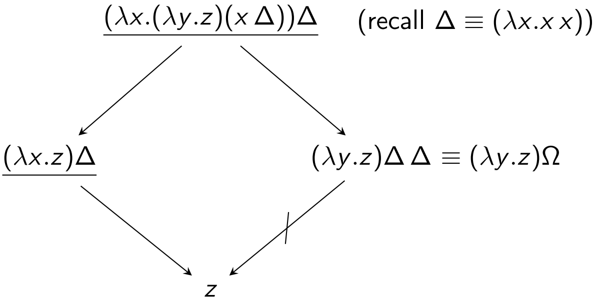
z



(recall $\Delta \equiv (\lambda x.x\,x)$)



(recall $\Delta \equiv (\lambda x.x\,x)$)


(λx.(λy.z)(x Δ))Δ
(recall Δ ≡ (λx.x x))
(λx.z)Δ
(λy.z)Δ Δ ≡ (λy.z)Ω
z

$$\mathbf{C}_{hs}[\,] \quad ::= \quad [\,] \mid \mathbf{C}_{hs}[\,]\Lambda \mid \lambda x.\mathbf{C}_{hs}[\,]$$

$$\begin{array}{lcl} \mathbf{C}_{hr}[\,] & ::= & [\,] \mid \mathbf{C}_{bn}[\,]\Lambda \mid \lambda x.\mathbf{C}_{hr}[\,] \\ \mathbf{C}_{bn}[\,] & ::= & [\,] \mid \mathbf{C}_{bn}[\,]\Lambda \end{array}$$

$$
\begin{aligned}
\mathbf{C}_{rc}[\,] &::= [\,] \mid \Lambda\,\mathbf{C}_{bv}[\,] \mid \mathbf{C}_{rc}[\,]\,\mathsf{WNF_V} \mid \lambda x.\mathbf{C}_{rc}[\,] \\
\mathbf{C}_{bv}[\,] &::= [\,] \mid \Lambda\,\mathbf{C}_{bv}[\,] \mid \mathbf{C}_{bv}[\,]\,\mathsf{WNF_V}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{C}_{ch}[\,] &::= [\,] \mid \Lambda\,\mathbf{C}_{bv}[\,] \mid \mathbf{C}_{bv}[\,]\,\mathsf{WNF_V} \mid \lambda x.\mathbf{C}_{ch}[\,] \\
\mathbf{C}_{bv}[\,] &::= [\,] \mid \Lambda\,\mathbf{C}_{bv}[\,] \mid \mathbf{C}_{bv}[\,]\,\mathsf{WNF_V}
\end{aligned}
$$

Completeness in λK

Any strategy which eventually contracts the redices in the head of the active components of a term is complete with respect to normal form.

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Reciprocally. . .

Theorem

Any strategy which is complete with respect to normal form eventually contracts the leftmost redex.

(It turns out that the leftmost redex is in positive position with respect to the leftmost active component, and hence it is not discardable!)

Completeness in λ_V

Theorem (Hereditary Quasi-Chest Reduction)

Any λ_V-strategy which eventually contracts the redices in the chest of the active components of a term is complete with respect to λ_V normal form.

Completeness in λ_V

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Any λ_V-strategy which eventually contracts the redices in the chest of the active components of a term is complete with respect to λ_V normal form.

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Theorem

Any λ_V-strategy which is complete with respect to λ_V normal form eventually contracts the redices in the chest of the active components of a term.