No solvable lambda-value term left behind

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Computable functions and pure lambda calculus

$$
\begin{aligned}
f &\in \mathbb{N} \to \mathbb{N} \\
f &= \{ (0,0), (1,2), (2,4), (3,6), (4,8), \ldots \} \\
f &= \{ (x,y) \mid y = x + x \} \\
f(x) &= x + x
\end{aligned}
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$$M, N ::= \; x \mid \lambda x.M \mid M\,N$$

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f &\equiv \; \lambda x.\, SUBS\, (MULT\, (ADD\, x\, ONE)\, TWO)\, TWO
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$$f(2) \equiv (\lambda x.ADD\, x\, x)\, TWO \longrightarrow ADD\, TWO\, TWO \longrightarrow \ldots$$

$$\begin{array}{lll} \mathbf{I} & \text{is} & \lambda x.\, x \\ \mathbf{K} & \text{is} & \lambda x.\, \lambda y.\, x \\ \mathbf{\Delta} & \text{is} & \lambda x.\, x\, x \\ \mathbf{\Omega} & \text{is} & \mathbf{\Delta}\,\mathbf{\Delta} \;\text{ is }\; (\lambda x.\, x\, x)(\lambda x.\, x\, x) \end{array}$$

$$\begin{array}{r} \mathbf{I}\, M \longrightarrow M \\ \mathbf{K}\, M\, N \longrightarrow (\lambda y.M)N \longrightarrow M \\ \mathbf{\Delta}\, M \longrightarrow M\, M \\ \mathbf{\Omega} \longrightarrow \mathbf{\Omega} \longrightarrow \mathbf{\Omega} \longrightarrow \dots \end{array}$$

Reduction in λ

$$\frac{}{(\lambda x.B)\,N \longrightarrow [N/x]B}\ (\beta)$$

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M has normal form : $M \longrightarrow Z$ and Z is a normal form

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Conversion in λ

$$\frac{}{(\lambda x.B)\,N \;=\; [N/x]B}\,(\beta)$$

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M is a normal form	:	no $(\lambda x.B)N$ subterm in M
M has normal form	:	$M = Z$ and Z is a normal form
Z unique	:	consistent proof theory

Can all terms without normal form be considered equal?

$$\mathbf{\Omega} = \lambda x.x\,\mathbf{\Omega}$$

iff

$$\mathbf{\Omega}\,(\mathbf{K}\,\mathbf{I}) = (\lambda x.x\,\mathbf{\Omega})(\mathbf{K}\,\mathbf{I})$$

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Adding axioms that equate terms without normal form:
inconsistent proof theory!

Solvability

Definition (Barendregt '71)

M closed is solvable iff there exist closed operands $N_1, \ldots, N_n$ with $n \geq 0$ s.t. $M\, N_1 \cdots N_n = Z$ a normal form.

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If T closed has normal form then for every term X there exist $X_1, \ldots, X_k$ s.t. $T\, X_1 \cdots X_k = X$.

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Definition (Barendregt '84)

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Solvability (open terms)

Definition (Wadsworth '78)

M arbitrary is solvable iff there exists $(\lambda x_1 \ldots x_m.[\,])N_1 \cdots N_n$ with $m, n \geq 0$ s.t.

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$$(\lambda x.[x\,\mathbf{\Omega}])(\mathbf{K}\,\mathbf{I}) \longrightarrow \mathbf{K}\,\mathbf{I}\,\mathbf{\Omega} \longrightarrow \mathbf{I}$$

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Operational relevance and effective use

Genericity Lemma (Barendregt '84)

M unsolvable then

$$\forall \mathcal{C}[\ ].\, \mathcal{C}[M] = Z \text{ a normal form} \Rightarrow \forall X.\, \mathcal{C}[X] = Z.$$

Operational relevance and effective use

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normal forms $\subset$ solvables

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normal forms $\subset$ solvables

Adding axioms that equate unsolvable terms:
consistent proof theory!
Full reduction and open terms: suitable for metaprogramming!

Conversion in the lambda-value calculus (λ_V)

(Plotkin '75)

$$\frac{N \in \mathsf{Val} ::= \; x \mid \lambda x.M}{(\lambda x.B)\, N \;\; =_V \;\; [N/x]B} \; (\beta_V)$$

$$\frac{M \;\; =_V \;\; M'}{M\, N \;\; =_V \;\; M'\, N} \qquad \frac{N \;\; =_V \;\; N'}{M\, N \;\; =_V \;\; M\, N'} \qquad \frac{B \;\; =_V \;\; B'}{\lambda x.B \;\; =_V \;\; \lambda x.B'}$$

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Why values instead of normal forms?

Preserving confluence by preserving potential divergence.

$$(\lambda x.(\lambda y.\mathbf{I})(x\,\mathbf{\Delta}))\mathbf{\Delta}$$

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Sequential character of λ_V-normal-forms.

$$(x\,M)(y\,N)$$
$$(\lambda z.z(y\,N))(x\,M)$$

v-solvability

Definition (Paolini & Ronchi della Rocca '99)

M closed is v-solvable iff there exist closed values $V_1, \ldots, V_n$ with $n \geq 0$ s.t. $M\, V_1 \cdots V_n =_V \mathbf{I}$.

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$$\longrightarrow (\lambda y.\boldsymbol{\Delta})\boldsymbol{\Delta} \longrightarrow \boldsymbol{\Omega} \longrightarrow \ldots$$

Does not capture operational relevance!

Adding axioms that equate v-unsolvables: inconsistent proof theory!

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SECD Machine (Landin '64).

Lazy call-by-value evaluation (Egidi, Honsell, Ronchi della Rocca '91).

Only weak reduction and closed terms: not suitable for metaprogramming!

Start from scratch

Definition (García-Pérez & Nogueira 2016)

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- Definition of λ_V-solvability that captures operational relevance of arbitrary terms (not necessarily closed).
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- Sequentiality of terms matters.
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Thanks!