No solvable lambda-value term left behind

Alvaro García Pérez and Pablo Nogueira

Reykjavik University IMDEA Software Institute

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Computable functions and pure lambda calculus

\[ f \in \mathbb{N} \rightarrow \mathbb{N} \]
\[ f = \{ (0, 0), (1, 2), (2, 4), (3, 6), (4, 8), \ldots \} \]
\[ f = \{ (x, y) \mid y = x + x \} \]
\[ f(x) = x + x \]
Computable functions and pure lambda calculus

\[ M, N ::= x \mid \lambda x. M \mid M N \]

\[ f \in \mathbb{N} \rightarrow \mathbb{N} \]

\[ f = \{ (0, 0), (1, 2), (2, 4), (3, 6), (4, 8), \ldots \} \]

\[ f = \{ (x, y) \mid y = x + x \} \]

\[ f(x) = x + x \]

\[ f \equiv \lambda x. ADD x x \]
Computable functions and pure lambda calculus

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\[ f(x) = x + x \]

\[ f \equiv \lambda x. ADD x x \quad ((ADD x_1) \equiv g \text{ where } g(x_2) = x_1 + x_2) \]
Computable functions and pure lambda calculus

\[ M, N ::= x | \lambda x. M | M N \]

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\[ f = \{ (x, y) \mid y = x + x \} \]

\[ f(x) = x + x \]

\[ f \equiv \lambda x. \text{ADD} \, x \, x \quad (\text{ADD} \, x_1) \equiv g \text{ where } g(x_2) = x_1 + x_2 \]

\[ f(2) \equiv (\lambda x. \text{ADD} \, x \, x) \, \text{TWO} \rightarrow \text{ADD} \, \text{TWO} \, \text{TWO} \rightarrow \ldots \]
Computable functions and pure lambda calculus

\[ M, N ::= x | \lambda x. M | M N \]

\[ f \in \mathbb{N} \rightarrow \mathbb{N} \]

\[ f = \{ (0, 0), (1, 2), (2, 4), (3, 6), (4, 8), \ldots \} \]

\[ f = \{ (x, y) | y = x + x \} \]

\[ f(x) = x + x \]

\[ f \equiv \lambda x. ADD \ x \ x \quad ((ADD \ x_1) \equiv g \text{ where } g(x_2) = x_1 + x_2) \]

\[ f \equiv \lambda x. MULT \ x \ TWO \]

\[ f \equiv \lambda x. SUBS (MULT (ADD \ x \ ONE) TWO) \ TWO \]

\[ f(2) \equiv (\lambda x. ADD \ x \ x) \ TWO \rightarrow ADD \ TWO \ TWO \rightarrow \ldots \]
\( \text{I is } \lambda x. x \) \\
\( \text{K is } \lambda x. \lambda y. x \)

\( \text{K M N} \rightarrow (\lambda y. M) N \rightarrow M \)

\( \text{Δ is } \lambda x. x x \)

\( \text{Δ M} \rightarrow M M \)

\( \text{Ω is } \Delta \Delta \text{ is } (\lambda x. x x)(\lambda x. x x) \)

\( \text{Ω} \rightarrow \Omega \rightarrow \Omega \rightarrow \Omega \rightarrow \ldots \)
Reduction in $\lambda$

$$\frac{(\lambda x. B) N \rightarrow [N/x]B}{(\beta)}$$
Reduction in $\lambda$

$$(\lambda x. B) \rightarrow [N/x] B \quad (\beta)$$

\[
\begin{align*}
M \rightarrow M' & \quad \frac{M N \rightarrow M' N}{M N \rightarrow M' N} \\
N \rightarrow N' & \quad \frac{M N \rightarrow M N'}{M N \rightarrow M N'} \\
B \rightarrow B' & \quad \frac{\lambda x. B \rightarrow \lambda x. B'}{\lambda x. B \rightarrow \lambda x. B'}
\end{align*}
\]
Reduction in $\lambda$

\[
(\lambda x. B) N \rightarrow [N/x]B \quad (\beta)
\]

\[
\begin{align*}
M & \rightarrow M' \\
M N & \rightarrow M' N \\
N & \rightarrow N' \\
MN & \rightarrow MN' \\
\lambda x. B & \rightarrow \lambda x. B'
\end{align*}
\]

\[
\begin{align*}
M & \rightarrow M \\
M & \rightarrow N \\
N & \rightarrow P \\
M & \rightarrow P
\end{align*}
\]
Reduction in $\lambda$

$$(\lambda x. B) N \rightarrow [N/x]B \tag{\beta}$$

$$
\begin{align*}
M & \rightarrow M' \\
\hline
M N & \rightarrow M' N
\end{align*}
$$

$$
\begin{align*}
N & \rightarrow N' \\
\hline
M N & \rightarrow M N'
\end{align*}
$$

$$
\begin{align*}
B & \rightarrow B' \\
\hline
\lambda x. B & \rightarrow \lambda x. B'
\end{align*}
$$

$$
\begin{align*}
M & \rightarrow N \\
\hline
N & \rightarrow P
\end{align*}
$$

$$
\begin{align*}
M & \rightarrow M \\
\hline
M & \rightarrow P
\end{align*}
$$

$M$ is a normal form : no $(\lambda x. B)N$ subterm in $M$

$M$ has normal form : $M \rightarrow Z$ and $Z$ is a normal form
Reduction in $\lambda$

$$(\lambda x. B) \ N \rightarrow [N/x]B \quad (\beta)$$

\[
\begin{align*}
M &\rightarrow M' & N &\rightarrow N' & B &\rightarrow B' \\
MN &\rightarrow M'N & MN &\rightarrow MN' & \lambda x. B &\rightarrow \lambda x. B' \\
M &\rightarrow M & M &\rightarrow N & N &\rightarrow P
\end{align*}
\]

$M$ is a normal form : no $(\lambda x. B)N$ subterm in $M$

$M$ has normal form : $M \rightarrow Z$ and $Z$ is a normal form

$Z$ unique
Conversion in $\lambda$

$$(\lambda x. B) \, N \ = \ [N/x]B \quad (\beta)$$

$$\begin{align*}
M & = M' \\
\frac{M \, N}{M' \, N}
\end{align*}$$

$$\begin{align*}
N & = N' \\
\frac{M \, N}{M \, N'}
\end{align*}$$

$$\begin{align*}
B & = B' \\
\frac{\lambda x. B}{\lambda x. B'}
\end{align*}$$

$$\begin{align*}
M & = N \\
\frac{M = N}{N = M}
\end{align*}$$

$M$ is a normal form : no $(\lambda x. B)N$ subterm in $M$

$M$ has normal form : $M = Z$ and $Z$ is a normal form

$Z$ unique : consistent proof theory
Can all terms without normal form be considered equal?

\[ \Omega = \lambda x. x \Omega \]

\[ \text{iff} \]

\[ \Omega(K I) = (\lambda x. x \Omega)(K I) \]

\[ \text{iff} \]

\[ \Omega(K I) = K I \Omega \]

\[ \text{iff} \]

\[ \Omega(K I) = I \]
Can all terms without normal form be considered equal?

\[
\Omega = \lambda x. x \Omega
\]

iff

\[
\Omega (K I) = (\lambda x. x \Omega)(K I)
\]

iff

\[
\Omega (K I) = K I \Omega
\]

iff

\[
\Omega (K I) = I
\]

Adding axioms that equate terms without normal form: inconsistent proof theory!
Solvability

Definition (Barendregt ’71)

\( M \) closed is solvable iff there exist closed operands \( N_1, \ldots, N_n \) with \( n \geq 0 \) s.t. \( M N_1 \cdots N_n = Z \) a normal form.
Solvability

Definition (Barendregt ’71)

\( M \) closed is solvable iff there exist closed operands \( N_1, \ldots, N_n \) with \( n \geq 0 \) s.t. \( M \ N_1 \cdots \ N_n = Z \) a normal form.

\( \Omega \) is unsolvable
\( \lambda x.x \Omega \) is solvable
Solvability

Definition (Barendregt '71)

$M$ closed is solvable iff there exist closed operands $N_1, \ldots, N_n$ with $n \geq 0$ s.t. $M \, N_1 \cdots N_n = Z$ a normal form.

Definition (Wadsworth '78)

$M$ closed is solvable iff for every term $X$ there exist closed operands $N_1, \ldots, N_n$ with $n \geq 0$ s.t. $M \, N_1 \cdots N_n = X$.

Lemma (Wadsworth '78)

If $T$ closed has normal form then for every term $X$ there exist $X_1, \ldots, X_k$ s.t. $T \, X_1 \cdots X_k = X$. 
Solvability

Definition (Barendregt '71)

$M$ closed is solvable iff there exist closed operands $N_1, \ldots, N_n$ with $n \geq 0$ s.t. $M N_1 \cdots N_n = Z$ a normal form.

Definition (Wadsworth '78)

$M$ closed is solvable iff for every term $X$ there exist closed operands $N_1, \ldots, N_n$ with $n \geq 0$ s.t. $M N_1 \cdots N_n = X$.

Lemma (Wadsworth '78)

If $T$ closed has normal form then for every term $X$ there exist $X_1, \ldots, X_k$ s.t. $T X_1 \cdots X_k = X$.

Definition (Barendregt '84)

$M$ closed is solvable iff there exist closed operands $N_1, \ldots, N_n$ with $n \geq 0$ s.t. $M N_1 \cdots N_n = I$. 
Solvability (open terms)

Definition (Wadsworth ’78)

$M$ arbitrary is solvable iff there exists $(\lambda x_1 \ldots x_m.[ ] )N_1 \cdots N_n$ with $m, n \geq 0$ s.t.

$$(\lambda x_1 \ldots x_m. [M] )N_1 \cdots N_n = Z \text{ a normal form.}$$
Solvability (open terms)

Definition (Wadsworth ’78)

\(M\) arbitrary is solvable iff there exists \((\lambda x_1 \ldots x_m.[\ ]))N_1 \ldots N_n\) with \(m, n \geq 0\) s.t.

\[
(\lambda x_1 \ldots x_m.[M])N_1 \ldots N_n = Z \text{ a normal form.}
\]
Solvability (open terms)

Definition (Wadsworth '78)

$M$ arbitrary is solvable iff there exists $(\lambda x_1 \ldots x_m.[\ ]){N_1} \cdots {N_n}$
with $m, n \geq 0$ s.t.

\[(\lambda x_1 \ldots x_m.[M]){N_1} \cdots {N_n} = Z\] a normal form.

\[(\lambda x.[x \Omega])(K I)\]
Solvability (open terms)

Definition (Wadsworth ’78)

\(M\) arbitrary is solvable iff there exists \((\lambda x_1 \ldots x_m.[\ ]\))\(N_1 \cdots N_n\) with \(m, n \geq 0\) s.t.

\[
(\lambda x_1 \ldots x_m[M])N_1 \cdots N_n = Z \text{ a normal form.}
\]

\[
(\lambda x.[x \Omega])(K I) \rightarrow K I \Omega \rightarrow I
\]
Solvability (open terms)

Definition (Wadsworth ’78)

$M$ arbitrary is solvable iff for all $X$ there exists
$(\lambda x_1 \ldots x_m.[\ ])_N \cdot N_n$ with $m, n \geq 0$ s.t.

$$(\lambda x_1 \ldots x_m.[M])N_1 \cdots N_n = X.$$  

$$(\lambda x.[x \Omega])(KI) \rightarrow KI\Omega \rightarrow I$$
Solvability (open terms)

Definition (Wadsworth ’78)

$M$ arbitrary is solvable iff there exists $(\lambda x_1 \ldots x_m.[ ] ) N_1 \cdots N_n$
with $m, n \geq 0$ s.t.

$$(\lambda x_1 \ldots x_m. [M ] ) N_1 \cdots N_n = I.$$

$$(\lambda x. [x \Omega]) (K I) \rightarrow K I \Omega \rightarrow I$$
Operational relevance and effective use

Genericity Lemma (Barendregt ’84)

$M$ unsolvable then

$$\forall C[ ]. C[M] = Z \text{ a normal form } \Rightarrow \forall X. C[X] = Z.$$
Operational relevance and effective use

Genericity Lemma (Barendregt ’84)

$M$ unsolvable iff

$$\forall C[\ ]. C[M] = Z \text{ a normal form } \Rightarrow \forall X. C[X] = Z.$$
Operational relevance and effective use

Genericity Lemma (Barendregt ’84)

$M$ is solvable iff

$$\exists C\ [\ ].\ C[M] = Z \text{ a normal form } \land \neg(\forall X. C[X] = Z).$$

Adding axioms that equate unsolvable terms:

consistent proof theory!

Full reduction and open terms: suitable for metaprogramming!
Operational relevance and effective use

Genericity Lemma (Barendregt ’84)

\( M \) is solvable iff

\[ \exists \mathcal{C}[ \cdot]. \mathcal{C}[M] = Z \text{ a normal form } \land \lnot (\forall X. \mathcal{C}[X] = Z). \]

normal forms \( \subset \) solvables
Operational relevance and effective use

Genericity Lemma (Barendregt ’84)

$M$ is solvable iff

$$\exists C[.] \cdot C[M] = Z \text{ a normal form} \land \neg (\forall X. C[X] = Z).$$

Normal forms $\subset$ solvables

Adding axioms that equate unsolvable terms:
consistent proof theory!
Full reduction and open terms: suitable for metaprogramming!
Conversion in the lambda-value calculus ($\lambda_V$)

(Plotkin '75)

\[
\begin{align*}
N \in \text{Val} & ::= \ x \mid \lambda x. M \\
(\lambda x. B) N & =_V \ [N/x]B & (\beta_V)
\end{align*}
\]

\[
\begin{align*}
M & =_V M' \quad & N & =_V N' & B & =_V B' \\
MN & =_V M'N & MN & =_V MN' & \lambda x. B & =_V \lambda x. B' \\
M & =_V M & M =_V N & N =_V P & M =_V N \\
& & M =_V P & N =_V M
\end{align*}
\]
Why values instead of normal forms?

Preserving confluence by preserving potential divergence.

\[(\lambda x. (\lambda y. I)(x \Delta)) \Delta\]
Why values instead of normal forms?

Preserving confluence by preserving potential divergence.

$$(\lambda x. (\lambda y. I)(x \Delta)) \Delta$$
Why values instead of normal forms?

Preserving confluence by preserving potential divergence.

\[(\lambda x. (\lambda y. I)(x \Delta))(\Delta)\]
Why values instead of normal forms?

Preserving confluence by preserving potential divergence.

\[(\lambda x. (\lambda y. I)(x \Delta))\Delta \rightarrowV (\lambda x. I)\Delta\]
Why values instead of normal forms?

Preserving confluence by preserving potential divergence.

\[(\lambda x. (\lambda y. I)(x \Delta))\Delta \rightarrow_V (\lambda x. I)\Delta \rightarrow_V I\]
Why values instead of normal forms?

Preserving confluence by preserving potential divergence.

\[
(\lambda x.(\lambda y.I)(x \Delta))\Delta \twoheadrightarrow^n (\lambda x.I)\Delta \twoheadrightarrow^n I
\]

\[
(\lambda x.(\lambda y.I)(x \Delta))\Delta
\]
Why values instead of normal forms?

Preserving confluence by preserving potential divergence.

\[(\lambda x.(\lambda y.I)(x \Delta))\Delta \rightarrow^V (\lambda x.I)\Delta \rightarrow^V I\]

\[(\lambda x.(\lambda y.I)(x \Delta))\Delta \rightarrow^V (\lambda y.I)(\Delta \Delta)\]
Why values instead of normal forms?

Preserving confluence by preserving potential divergence.

$$(\lambda x. (\lambda y. I)(x \Delta))\Delta \rightarrow^v (\lambda x. I)\Delta \rightarrow^v I$$

$$(\lambda x. (\lambda y. I)(x \Delta))\Delta \rightarrow^v (\lambda y. I)\Omega$$

$$(\lambda x. (\lambda y. I)(x \Delta))\Delta \rightarrow^v (\lambda y. I)\Omega$$
Why values instead of normal forms?

Preserving confluence by preserving potential divergence.

\[
\begin{align*}
(\lambda x. (\lambda y. I)(x \Delta))\Delta & \rightarrow_V (\lambda x. I)\Delta \rightarrow_V I \\
(\lambda x. (\lambda y. I)(x \Delta))\Delta & \rightarrow_V (\lambda y. I)\Omega \rightarrow_V \ldots
\end{align*}
\]
Why values instead of normal forms?

Preserving confluence by preserving potential divergence.

\[
(\lambda x.(\lambda y.1)(x \Delta))\Delta \rightarrow_V (\lambda x.1)\Delta \rightarrow_V 1
\]

\[
(\lambda x.(\lambda y.1)(x \Delta))\Delta \rightarrow_V (\lambda y.1)\Omega \rightarrow_V \ldots
\]
Why values instead of normal forms?

Preserving confluence by preserving potential divergence.

\[(\lambda x. (\lambda y. I)(x \Delta)) \Delta \rightarrow^V (\lambda x. I) \Delta \rightarrow^V I\]

\[(\lambda x. (\lambda y. I)(x \Delta)) \Delta \rightarrow^V (\lambda y. I) \Omega \rightarrow^V \ldots\]

\(\lambda_V\)-normal-forms don’t have subterms \((\lambda x. B)N\) with \(N \in Val\).

\((\lambda y. I)(x \Delta)\) is a \(\lambda_V\)-normal-form
Why values instead of normal forms?

Preserving confluence by preserving potential divergence.

\[
(\lambda x.(\lambda y.l)(x \Delta))\Delta \longrightarrow_V (\lambda x.l)\Delta \longrightarrow_V l
\]

\[
(\lambda x.(\lambda y.l)(x \Delta))\Delta \longrightarrow_V (\lambda y.l)\Omega \longrightarrow_V \ldots
\]

\(\lambda_V\)-normal-forms don’t have subterms \((\lambda x.B)N\) with \(N \in \text{Val}\).

\((\lambda y.l)(x \Delta)\) is a \(\lambda_V\)-normal-form

Sequential character of \(\lambda_V\)-normal-forms.

\[(x M)(y N)\]
\[(\lambda z.z(y N))(x M)\]
**v-solvability**

**Definition (Paolini & Ronchi della Rocca ’99)**

\[ M \text{ closed is } v\text{-solvable iff there exist closed values } V_1, \ldots, V_n \text{ with } n \geq 0 \text{ s.t. } M \, V_1 \cdots \, V_n =_V \, I. \]
**ν-solvability**

**Definition (Paolini & Ronchi della Rocca '99)**

\( M \text{ closed is } \nu\text{-solvable} \) iff there exist closed values \( V_1, \ldots, V_n \) with \( n \geq 0 \) s.t. \( M V_1 \cdots V_n =_\nu \mathbf{I} \).

**Lemma (analogous to Wadsworth '78)**

If \( T \text{ closed has } \lambda_\nu\text{-normal-form} \) then for every term \( X \) there exist values \( V_1, \ldots, V_k \) s.t. \( T V_1 \cdots V_k =_\nu X \).
**ν-solvability**

**Definition (Paolini & Ronchi della Rocca ’99)**

$M$ **closed** is **ν-solvable** iff there exist **closed** values $V_1, \ldots, V_n$ with $n \geq 0$ s.t. $M \ V_1 \cdots V_n =_V I$.

**Lemma (analogous to Wadsworth ’78)**

If $T$ **closed** has $\lambda V$-normal-form then for every term $X$ there exist values $V_1, \ldots, V_k$ s.t. $T \ V_1 \cdots V_k =_V X$.

Some $\lambda V$-normal-forms are **ν-unsolvable**!

$$\lambda x. (\lambda y. \Delta)(x \ I) \Delta$$
**ν-solvability**

**Definition (Paolini & Ronchi della Rocca ’99)**

\( M \text{ closed is } \nu\text{-solvable iff there exist closed values } V_1, \ldots, V_n \) with \( n \geq 0 \) s.t. \( M \; V_1 \cdots V_n =_V I \).

**Lemma (analogous to Wadsworth ’78)**

If \( T \text{ closed has } \lambda_V\text{-normal-form} \) then for every term \( X \) there exist values \( V_1, \ldots, V_k \) s.t. \( T \; V_1 \cdots V_k =_V X \).

Some \( \lambda_V\)-normal-forms are \( \nu\)-unsolvable!

\[
\left( \lambda x. (\lambda y. \Delta)(x \; I)\Delta \right) V_1
\]
**ν-solvability**

**Definition (Paolini & Ronchi della Rocca ’99)**

$M \text{ closed}$ is $ν$-solvable iff there exist closed values $V_1, \ldots, V_n$ with $n \geq 0$ s.t. $M V_1 \cdots V_n =_V I$.

**Lemma (analogous to Wadsworth ’78)**

If $T \text{ closed}$ has $λ_ν$-normal-form then for every term $X$ there exist values $V_1, \ldots, V_k$ s.t. $T V_1 \cdots V_k =_V X$.

Some $λ_ν$-normal-forms are $ν$-unsolvable!

$$(λx.(λy.Δ)(x I)Δ)V_1 \rightarrow_ν (λy.Δ)(V_1 I)Δ$$
**v-solvability**

**Definition (Paolini & Ronchi della Rocca ’99)**

$M$ **closed** is **v-solvable** iff there exist **closed values** $V_1, \ldots, V_n$ with $n \geq 0$ s.t. $M \, V_1 \cdots V_n =_V I$.

**Lemma (analogous to Wadsworth ’78)**

If $T$ **closed** has $\lambda_V$-normal-form then for every term $X$ there exist **values** $V_1, \ldots, V_k$ s.t. $T \, V_1 \cdots V_k =_V X$.

Some $\lambda_V$-normal-forms are **v-unsolvable**!

$$(\lambda x. (\lambda y. \Delta)(x \, I)\Delta) \, V_1 \longrightarrow_V (\lambda y. \Delta)(V_1 \, I)\Delta$$
\textbf{v-solvability}

**Definition (Paolini & Ronchi della Rocca ’99)**

\( M \) closed is **v-solvable** iff there exist closed values \( V_1, \ldots, V_n \) with \( n \geq 0 \) s.t. \( M \, V_1 \cdots V_n =_v I \).

**Lemma (analogous to Wadsworth ’78)**

If \( T \) closed has \( \lambda_V \)-normal-form then for every term \( X \) there exist values \( V_1, \ldots, V_k \) s.t. \( T \, V_1 \cdots V_k =_V X \).

Some \( \lambda_V \)-normal-forms are \( v \)-unsolvable!

\[
(\lambda x. (\lambda y. \Delta)(x \, I) \Delta) \, V_1 \rightarrow_v (\lambda y. \Delta)(V_1 \, I) \Delta \rightarrow \ldots
\]
\( \nu \)-solvability

**Definition (Paolini & Ronchi della Rocca ’99)**

\( M \) closed is \( \nu \)-solvable iff there exist closed values \( V_1, \ldots, V_n \) with \( n \geq 0 \) s.t. \( M \ V_1 \cdots V_n =_V \ I \).

**Lemma (analogous to Wadsworth ’78)**

If \( T \) closed has \( \lambda_V \)-normal-form then for every term \( X \) there exist values \( V_1, \ldots, V_k \) s.t. \( T \ V_1 \cdots V_k =_V X \).

Some \( \lambda_V \)-normal-forms are \( \nu \)-unsolvable!

\[
(\lambda x.(\lambda y.\Delta)(x \ I)\Delta) \ V_1 \longrightarrow_V (\lambda y.\Delta)(V_1 \ I)\Delta
\]
**v-solvability**

**Definition (Paolini & Ronchi della Rocca ’99)**

$M$ closed is \( v \text{-solvable} \) iff there exist \( \text{closed values} \) \( V_1, \ldots, V_n \) with \( n \geq 0 \) s.t. \( M \ V_1 \cdots V_n =_v I \).

**Lemma (analogous to Wadsworth ’78)**

If \( T \) closed has \( \lambda \nu \)-normal-form then for every term \( X \) there exist \( \text{values} \) \( V_1, \ldots, V_k \) s.t. \( T \ V_1 \cdots V_k =_v X \).

Some \( \lambda \nu \)-normal-forms are \( v \)-unsolvable!

\[
(\lambda x. (\lambda y. \Delta)(x \ I) \Delta) \ V_1 \rightarrow_v (\lambda y. \Delta)(V_1 \ I) \Delta \\
\rightarrow (\lambda y. \Delta) \Delta
\]
\[ v \text{-solvability} \]

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**Lemma (analogous to Wadsworth '78)**

If \( T \) closed has \( \lambda_V \)-normal-form then for every term \( X \) there exist values \( V_1, \ldots, V_k \) s.t. \( T \, V_1 \cdots V_k =_V X \).

Some \( \lambda_V \)-normal-forms are \( v \)-unsolvable!

\[
(\lambda x.(\lambda y.\Delta)(x \, I)\Delta) \, V_1 \rightarrow^v (\lambda y.\Delta)(V_1 \, I)\Delta
\]

\[
\rightarrow (\lambda y.\Delta)\Delta \rightarrow \Delta\Delta
\]
$\nu$-solvability

**Definition (Paolini & Ronchi della Rocca ’99)**

$M$ closed is $\nu$-solvable iff there exist closed values $V_1, \ldots, V_n$ with $n \geq 0$ s.t. $M V_1 \cdots V_n =_\nu I$.

**Lemma (analogous to Wadsworth ’78)**

If $T$ closed has $\lambda_\nu$-normal-form then for every term $X$ there exist values $V_1, \ldots, V_k$ s.t. $T V_1 \cdots V_k =_\nu X$.

Some $\lambda_\nu$-normal-forms are $\nu$-unsolvable!

\[
(\lambda x. (\lambda y. \Delta)(x \ I)\Delta) V_1 \rightarrow_\nu (\lambda y. \Delta)(V_1 \ I)\Delta \\
\rightarrow (\lambda y. \Delta)\Delta \rightarrow \Omega \rightarrow \ldots
\]
**v-solvability**

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Some $\lambda_V$-normal-forms are $v$-unsolvable!

$$
(\lambda x.(\lambda y.\Delta)(x \ I)\Delta) \ V_1 \rightarrow_V (\lambda y.\Delta)(V_1 \ I)\Delta \\
\quad \rightarrow (\lambda y.\Delta)\Delta \rightarrow \Omega \rightarrow \ldots
$$

Does not capture operational relevance!

Adding axioms that equate $v$-unsolvables: inconsistent proof theory!
\textbf{\textit{v}-solvability}

**Definition (Paolini & Ronchi della Rocca '99)**

\(M\text{ \underline{closed}}\text{ is } v\text{-solvable} \iff \text{there exist closed values } V_1, \ldots, V_n \text{ with } n \geq 0 \text{ s.t. } \ M \ V_1 \cdots V_n =_v \text{ I}.

SECD Machine (Landin '64).

Lazy call-by-value evaluation (Egidi, Honsell, Ronchi della Rocca '91).

Only weak reduction and closed terms: not suitable for metaprogramming!
Start from scratch

Definition (García-Pérez & Nogueira 2016)

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The $\lambda_V$-normal forms are $\lambda_V$-solvable!
Adding axioms equating $\lambda_V$-unsolvables of the same order:
consistent proof theory!
Full reduction and open terms: suitable for metaprogramming!
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- Transformability. Ability to send a term to a value of your choice (captured by \( \nu \)-solvability too).
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Our contributions

- Definition of $\lambda_V$-solvability that captures operational relevance of arbitrary terms (not necessarily closed).
- Full reduction and open terms.
- Sequentiality of terms matters.
- Consistent proof theory.
- Partial Genericity Lemma (order of a term matters).
- Characterisation of complete reduction strategies with respect to $\lambda_V$-normal-form.
- Do models for sequentiality (Berry & Curien ’82) satisfy our theory?
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Thanks!