Deriving the Full-Reducing Krivine Machine from the Small-Step Operational Semantics of Normal Order

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Derivational Diagram

$$
\begin{tikzcd}
\lambda & \lambda_{\widetilde{\rho}} & \lambda^{*}_{\overline{\rho}} & \begin{array}{c}\text{exp. ctrl.}\\ \lambda^{*}_{\overline{\rho}}\end{array} \\
\begin{array}{c}\text{red.-based}\\ \text{normaliser}\end{array} & \begin{array}{c}\textbf{red.-based}\\ \textbf{normaliser}\end{array} \arrow[l, "\sigma\text{c}"'] \arrow[d, "\begin{array}{l}\text{refocus}\\ +\ \text{LWF}\\ +\ \text{inline}\end{array}"] & \begin{array}{c}\textbf{push/enter}\\ \textbf{KN}\end{array} & \\
 & \begin{array}{c}\text{eval/apply}\\ \text{abs. mach.}\end{array} \arrow[d, "\begin{array}{l}\text{refunc.}\\ +\ \text{CPS}^{-1}\end{array}"] & \begin{array}{c}\text{eval/apply}\\ \text{abs. mach.}\end{array} \arrow[u, "\begin{array}{l}\text{protrude}\\ +\ \text{inline}\end{array}"'] & \begin{array}{c}\text{eval/apply}\\ \text{abs. mach.}\end{array} \arrow[l, "\begin{array}{c}\text{rm.}\\ \text{ctrl.}\end{array}"'] \\
 & \begin{array}{c}\text{red.-free}\\ \text{normaliser}\end{array} \arrow[d, "\text{prepone}"] & & \\
 & \begin{array}{c}\text{preponed}\\ \text{red.-free}\\ \text{normaliser}\end{array} \arrow[r, "\text{shortcut}"] & \begin{array}{c}\text{preponed}\\ \text{red.-free}\\ \text{normaliser}\end{array} \arrow[r, "\begin{array}{c}\text{exp.}\\ \text{ctrl.}\end{array}"] & \begin{array}{c}\text{preponed}\\ \text{red.-free}\\ \text{normaliser}\end{array} \arrow[uu, "\begin{array}{l}\text{CPS}\\ +\ \text{defunc}\end{array}"']
\end{tikzcd}
$$

Derivational Diagram

$$
\begin{tikzcd}[column sep=large, row sep=large]
\lambda & \lambda_{\widetilde{\rho}} & \lambda^*_{\overline{\rho}} & \begin{array}{c}\text{exp. ctrl.}\\ \lambda^*_{\overline{\rho}}\end{array} \\
\begin{array}{c}\text{red.-based}\\ \text{normaliser}\end{array} & \begin{array}{c}\textbf{red.-based}\\ \textbf{normaliser}\end{array} \arrow[l, "\sigma\mathrm{c}"'] \arrow[d, "\begin{array}{l}\text{refocus}\\ +\ \text{LWF}\\ +\ \text{inline}\end{array}"] & \begin{array}{c}\textbf{push/enter}\\ \textbf{KN}\end{array} & \\
 & \begin{array}{c}\text{eval/apply}\\ \text{abs. mach.}\end{array} \arrow[d, "\begin{array}{l}\text{refunc.}\\ +\ \text{CPS}^{-1}\end{array}"] & \begin{array}{c}\text{eval/apply}\\ \text{abs. mach.}\end{array} \arrow[u, "\begin{array}{l}\text{protrude}\\ +\ \text{inline}\end{array}"'] & \begin{array}{c}\text{eval/apply}\\ \text{abs. mach.}\end{array} \arrow[l, "\begin{array}{c}\text{rm.}\\ \text{ctrl.}\end{array}"'] \\
 & \begin{array}{c}\text{red.-free}\\ \text{normaliser}\end{array} \arrow[d, "\text{prepone}"] & & \\
 & \begin{array}{c}\text{preponed}\\ \text{red.-free}\\ \text{normaliser}\end{array} \arrow[r, "\text{shortcut}"] & \begin{array}{c}\text{preponed}\\ \text{red.-free}\\ \text{normaliser}\end{array} \arrow[r, "\begin{array}{c}\text{exp.}\\ \text{ctrl.}\end{array}"] & \begin{array}{c}\text{preponed}\\ \text{red.-free}\\ \text{normaliser}\end{array} \arrow[uu, "\begin{array}{l}\text{CPS}\\ +\ \text{defunc}\end{array}"']
\end{tikzcd}
$$

Pure Untyped Lambda Calculus (λ)

$$\Lambda \quad ::= \quad n \mid \lambda.\Lambda \mid \Lambda\,\Lambda$$

Example

$$\lambda.0((\lambda.0)0)$$

Example

$$\lambda.0((\lambda.0)0)$$

Example

$$\lambda.0\underline{((\lambda.0)0)}$$

Example

$$\lambda.0\;0$$

Full-Reducing Krivine Machine ($\lambda_{\overline{\rho}}$) [Crégut, 2007]

$$
\begin{array}{lcl}
\mathsf{C} & ::= & \Lambda[\rho] \mid \overline{n} \mid \lfloor \Lambda, l \rfloor \\
\rho & ::= & \epsilon \mid \mathsf{C} : \rho \\
\\
S & ::= & \epsilon \mid \Lambda[\rho] : S \mid \lambda\!\!\lambda : S \mid \lfloor \Lambda, l \rfloor : S
\end{array}
$$

T	$\to$	$(\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ )$
$(\ (n+1)[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ n[\rho]\ ,\ S\ ,\ l\ )$
$(\ 0[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ \mathsf{C}\ ,\ S\ ,\ l\ )$
$(\ (M\ N)[\rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ )$	$\to$	$(\ B[N[\rho'] : \rho]\ ,\ S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ )$
$(\ \overline{n}\ ,\ S\ ,\ l\ )$	$\to$	$(\ \lfloor l-n, l \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor M, l' \rfloor\ ,\ N[\rho] : S\ ,\ l\ )$	$\to$	$(\ N[\rho]\ ,\ \lfloor M, l' \rfloor : S\ ,\ l'\ )$
$(\ \lfloor B, l' \rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ )$	$\to$	$(\ \lfloor \lambda.B, l' \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor N, l' \rfloor\ ,\ \lfloor M, l'' \rfloor : S\ ,\ l\ )$	$\to$	$(\ \lfloor M\ N, l'' \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor T, l' \rfloor\ ,\ \epsilon\ ,\ l\ )$	$\to$	T

Example

$$\lambda.0((\lambda.0)0)$$

T	$\rightarrow$	$(\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ )$
$(\ (n+1)[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ n[\rho]\ ,\ S\ ,\ l\ )$
$(\ 0[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \mathsf{C}\ ,\ S\ ,\ l\ )$
$(\ (M\ N)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ )$	$\rightarrow$	$(\ B[N[\rho'] : \rho]\ ,\ S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ )$
$(\ \overline{n}\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor l-n, l \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor M, l' \rfloor\ ,\ N[\rho] : S\ ,\ l\ )$	$\rightarrow$	$(\ N[\rho]\ ,\ \lfloor M, l' \rfloor : S\ ,\ l'\ )$
$(\ \lfloor B, l' \rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor \lambda.B, l' \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor N, l' \rfloor\ ,\ \lfloor M, l'' \rfloor : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor M\ N, l'' \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor T, l' \rfloor\ ,\ \epsilon\ ,\ l\ )$	$\rightarrow$	T

Example

$$\lambda.0((\lambda.0)0)$$

T	$\rightarrow$	$(\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ )$
$(\ (n+1)[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ n[\rho]\ ,\ S\ ,\ l\ )$
$(\ 0[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \mathsf{C}\ ,\ S\ ,\ l\ )$
$(\ (M\ N)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ )$	$\rightarrow$	$(\ B[N[\rho'] : \rho]\ ,\ S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ )$
$(\ \overline{n}\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor l-n, l \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor M, l' \rfloor\ ,\ N[\rho] : S\ ,\ l\ )$	$\rightarrow$	$(\ N[\rho]\ ,\ \lfloor M, l' \rfloor : S\ ,\ l'\ )$
$(\ \lfloor B, l' \rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor \lambda.B, l' \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor N, l' \rfloor\ ,\ \lfloor M, l'' \rfloor : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor M\ N, l'' \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor T, l' \rfloor\ ,\ \epsilon\ ,\ l\ )$	$\rightarrow$	T

Example

$$(\ \lambda.0((\lambda.0)0)[\epsilon] \ , \ \epsilon \ , \ 0 \ )$$

T	$\rightarrow$	$(\ T[\epsilon] \ , \ \epsilon \ , \ 0 \ )$
$(\ (n+1)[\mathsf{C} : \rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ n[\rho] \ , \ S \ , \ l \ )$
$(\ 0[\mathsf{C} : \rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ \mathsf{C} \ , \ S \ , \ l \ )$
$(\ (M\,N)[\rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ M[\rho] \ , \ N[\rho] : S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ N[\rho'] : S \ , \ l \ )$	$\rightarrow$	$(\ B[N[\rho'] : \rho] \ , \ S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ B[\overline{l+1} : \rho] \ , \ \lambda\!\!\lambda : S \ , \ l+1 \ )$
$(\ \overline{n} \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ \lfloor l-n, l \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor M, l' \rfloor \ , \ N[\rho] : S \ , \ l \ )$	$\rightarrow$	$(\ N[\rho] \ , \ \lfloor M, l' \rfloor : S \ , \ l' \ )$
$(\ \lfloor B, l' \rfloor \ , \ \lambda\!\!\lambda : S \ , \ l \ )$	$\rightarrow$	$(\ \lfloor \lambda.B, l' \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor N, l' \rfloor \ , \ \lfloor M, l'' \rfloor : S \ , \ l \ )$	$\rightarrow$	$(\ \lfloor M\,N, l'' \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor T, l' \rfloor \ , \ \epsilon \ , \ l \ )$	$\rightarrow$	T

Example

$$(\ \lambda.0((\lambda.0)0)[\epsilon] \ , \ \epsilon \ , \ 0 \ )$$

T	$\to$	$(\ T[\epsilon] \ , \ \epsilon \ , \ 0 \ )$
$(\ (n+1)[\mathsf{C} : \rho] \ , \ S \ , \ l \ )$	$\to$	$(\ n[\rho] \ , \ S \ , \ l \ )$
$(\ 0[\mathsf{C} : \rho] \ , \ S \ , \ l \ )$	$\to$	$(\ \mathsf{C} \ , \ S \ , \ l \ )$
$(\ (M \ N)[\rho] \ , \ S \ , \ l \ )$	$\to$	$(\ M[\rho] \ , \ N[\rho] : S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ N[\rho'] : S \ , \ l \ )$	$\to$	$(\ B[N[\rho'] : \rho] \ , \ S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ S \ , \ l \ )$	$\to$	$(\ B[\overline{l+1} : \rho] \ , \ \lambda\!\!\lambda : S \ , \ l+1 \ )$
$(\ \overline{n} \ , \ S \ , \ l \ )$	$\to$	$(\ \lfloor l-n, l \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor M, l' \rfloor \ , \ N[\rho] : S \ , \ l \ )$	$\to$	$(\ N[\rho] \ , \ \lfloor M, l' \rfloor : S \ , \ l' \ )$
$(\ \lfloor B, l' \rfloor \ , \ \lambda\!\!\lambda : S \ , \ l \ )$	$\to$	$(\ \lfloor \lambda.B, l' \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor N, l' \rfloor \ , \ \lfloor M, l'' \rfloor : S \ , \ l \ )$	$\to$	$(\ \lfloor M \ N, l'' \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor T, l' \rfloor \ , \ \epsilon \ , \ l \ )$	$\to$	T

Example

$$(\ 0((\lambda.0)0)[\overline{1}] \ , \ \lambda\!\!\lambda \ , \ 1 \ )$$

T	$\to$	$(\ T[\epsilon] \ , \ \epsilon \ , \ 0 \ )$
$(\ (n+1)[\mathsf{C} : \rho] \ , \ S \ , \ l \ )$	$\to$	$(\ n[\rho] \ , \ S \ , \ l \ )$
$(\ 0[\mathsf{C} : \rho] \ , \ S \ , \ l \ )$	$\to$	$(\ \mathsf{C} \ , \ S \ , \ l \ )$
$(\ (M\,N)[\rho] \ , \ S \ , \ l \ )$	$\to$	$(\ M[\rho] \ , \ N[\rho] : S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ N[\rho'] : S \ , \ l \ )$	$\to$	$(\ B[\overline{N[\rho']} : \rho] \ , \ S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ S \ , \ l \ )$	$\to$	$(\ B[\overline{l+1} : \rho] \ , \ \lambda\!\!\lambda : S \ , \ l+1 \ )$
$(\ \overline{n} \ , \ S \ , \ l \ )$	$\to$	$(\ \lfloor l-n, l \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor M, l' \rfloor \ , \ N[\rho] : S \ , \ l \ )$	$\to$	$(\ N[\rho] \ , \ \lfloor M, l' \rfloor : S \ , \ l' \ )$
$(\ \lfloor B, l' \rfloor \ , \ \lambda\!\!\lambda : S \ , \ l \ )$	$\to$	$(\ \lfloor \lambda.B, l' \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor N, l' \rfloor \ , \ \lfloor M, l'' \rfloor : S \ , \ l \ )$	$\to$	$(\ \lfloor M\,N, l'' \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor T, l' \rfloor \ , \ \epsilon \ , \ l \ )$	$\to$	T

Example

$$(0((\lambda.0)0)[\overline{1}] \,,\ \lambda\!\!\lambda \,,\ 1)$$

T	$\to$	$(T[\epsilon] , \epsilon , 0)$
$((n+1)[\mathsf{C} : \rho] , S , l)$	$\to$	$(n[\rho] , S , l)$
$(0[\mathsf{C} : \rho] , S , l)$	$\to$	(C , S , l)
$((M\,N)[\rho] , S , l)$	$\to$	$(M[\rho] , N[\rho] : S , l)$
$((\lambda.B)[\rho] , N[\rho'] : S , l)$	$\to$	$(B[N[\rho'] : \rho] , S , l)$
$((\lambda.B)[\rho] , S , l)$	$\to$	$(B[\overline{l+1} : \rho] , \lambda\!\!\lambda : S , l+1)$
$(\overline{n} , S , l)$	$\to$	$(\lfloor l-n, l \rfloor , S , l)$
$(\lfloor M, l' \rfloor , N[\rho] : S , l)$	$\to$	$(N[\rho] , \lfloor M, l' \rfloor : S , l')$
$(\lfloor B, l' \rfloor , \lambda\!\!\lambda : S , l)$	$\to$	$(\lfloor \lambda.B, l' \rfloor , S , l)$
$(\lfloor N, l' \rfloor , \lfloor M, l'' \rfloor : S , l)$	$\to$	$(\lfloor M\,N, l'' \rfloor , S , l)$
$(\lfloor T, l' \rfloor , \epsilon , l)$	$\to$	T

Example

$$(\; 0[\bar{1}] \; , \; ((\lambda.0)0)[\bar{1}] : \lambda\!\!\lambda \; , \; 1 \;)$$

T	$\to$	$(\; T[\epsilon] \; , \; \epsilon \; , \; 0 \;)$
$(\; (n+1)[\mathsf{C} : \rho] \; , \; S \; , \; l \;)$	$\to$	$(\; n[\rho] \; , \; S \; , \; l \;)$
$(\; 0[\mathsf{C} : \rho] \; , \; S \; , \; l \;)$	$\to$	$(\; \mathsf{C} \; , \; S \; , \; l \;)$
$(\; (M\,N)[\rho] \; , \; S \; , \; l \;)$	$\to$	$(\; M[\rho] \; , \; N[\rho] : S \; , \; l \;)$
$(\; (\lambda.B)[\rho] \; , \; N[\rho'] : S \; , \; l \;)$	$\to$	$(\; B[N[\rho'] : \rho] \; , \; S \; , \; l \;)$
$(\; (\lambda.B)[\rho] \; , \; S \; , \; l \;)$	$\to$	$(\; B[\overline{l+1} : \rho] \; , \; \lambda\!\!\lambda : S \; , \; l+1 \;)$
$(\; \bar{n} \; , \; S \; , \; l \;)$	$\to$	$(\; \lfloor l-n, l \rfloor \; , \; S \; , \; l \;)$
$(\; \lfloor M, l' \rfloor \; , \; N[\rho] : S \; , \; l \;)$	$\to$	$(\; N[\rho] \; , \; \lfloor M, l' \rfloor : S \; , \; l' \;)$
$(\; \lfloor B, l' \rfloor \; , \; \lambda\!\!\lambda : S \; , \; l \;)$	$\to$	$(\; \lfloor \lambda.B, l' \rfloor \; , \; S \; , \; l \;)$
$(\; \lfloor N, l' \rfloor \; , \; \lfloor M, l'' \rfloor : S \; , \; l \;)$	$\to$	$(\; \lfloor M\,N, l'' \rfloor \; , \; S \; , \; l \;)$
$(\; \lfloor T, l' \rfloor \; , \; \epsilon \; , \; l \;)$	$\to$	T

Example

$$(\; 0[\overline{1}] \; , \; ((\lambda.0)0)[\overline{1}] : \lambda\!\!\lambda \; , \; 1 \;)$$

$$
\begin{array}{|rcl|}
\hline
T & \to & (\; T[\epsilon] \; , \; \epsilon \; , \; 0 \;) \\
\hline
(\; (n+1)[\mathsf{C} : \rho] \; , \; S \; , \; l \;) & \to & (\; n[\rho] \; , \; S \; , \; l \;) \\
(\; 0[\mathsf{C} : \rho] \; , \; S \; , \; l \;) & \to & (\; \mathsf{C} \; , \; S \; , \; l \;) \\
(\; (M\,N)[\rho] \; , \; S \; , \; l \;) & \to & (\; M[\rho] \; , \; N[\rho] : S \; , \; l \;) \\
(\; (\lambda.B)[\rho] \; , \; N[\rho'] : S \; , \; l \;) & \to & (\; B[N[\rho'] : \rho] \; , \; S \; , \; l \;) \\
(\; (\lambda.B)[\rho] \; , \; S \; , \; l \;) & \to & (\; B[\overline{l+1} : \rho] \; , \; \lambda\!\!\lambda : S \; , \; l+1 \;) \\
(\; \overline{n} \; , \; S \; , \; l \;) & \to & (\; \lfloor l-n, l \rfloor \; , \; S \; , \; l \;) \\
(\; \lfloor M, l' \rfloor \; , \; N[\rho] : S \; , \; l \;) & \to & (\; N[\rho] \; , \; \lfloor M, l' \rfloor : S \; , \; l' \;) \\
(\; \lfloor B, l' \rfloor \; , \; \lambda\!\!\lambda : S \; , \; l \;) & \to & (\; \lfloor \lambda.B, l' \rfloor \; , \; S \; , \; l \;) \\
(\; \lfloor N, l' \rfloor \; , \; \lfloor M, l'' \rfloor : S \; , \; l \;) & \to & (\; \lfloor M\,N, l'' \rfloor \; , \; S \; , \; l \;) \\
\hline
(\; \lfloor T, l' \rfloor \; , \; \epsilon \; , \; l \;) & \to & T \\
\hline
\end{array}
$$

Example

$$(\overline{1} \;,\; ((\lambda.0)0)[\overline{1}] : \lambda\!\!\lambda \;,\; 1)$$

$$\begin{array}{|rcl|}
\hline
T & \rightarrow & (T[\epsilon] \;,\; \epsilon \;,\; 0) \\
\hline
((n+1)[\mathsf{C} : \rho] \;,\; S \;,\; l) & \rightarrow & (n[\rho] \;,\; S \;,\; l) \\
(0[\mathsf{C} : \rho] \;,\; S \;,\; l) & \rightarrow & (\mathsf{C} \;,\; S \;,\; l) \\
((M\,N)[\rho] \;,\; S \;,\; l) & \rightarrow & (M[\rho] \;,\; N[\rho] : S \;,\; l) \\
((\lambda.B)[\rho] \;,\; N[\rho'] : S \;,\; l) & \rightarrow & (B[N[\rho'] : \rho] \;,\; S \;,\; l) \\
((\lambda.B)[\rho] \;,\; S \;,\; l) & \rightarrow & (B[\overline{l+1} : \rho] \;,\; \lambda\!\!\lambda : S \;,\; l+1) \\
(\overline{n} \;,\; S \;,\; l) & \rightarrow & (\lfloor l-n, l \rfloor \;,\; S \;,\; l) \\
(\lfloor M, l' \rfloor \;,\; N[\rho] : S \;,\; l) & \rightarrow & (N[\rho] \;,\; \lfloor M, l' \rfloor : S \;,\; l') \\
(\lfloor B, l' \rfloor \;,\; \lambda\!\!\lambda : S \;,\; l) & \rightarrow & (\lfloor \lambda.B, l' \rfloor \;,\; S \;,\; l) \\
(\lfloor N, l' \rfloor \;,\; \lfloor M, l'' \rfloor : S \;,\; l) & \rightarrow & (\lfloor M\,N, l'' \rfloor \;,\; S \;,\; l) \\
\hline
(\lfloor T, l' \rfloor \;,\; \epsilon \;,\; l) & \rightarrow & T \\
\hline
\end{array}$$

Example

$$(\overline{1} \;,\; ((\lambda.0)0)[\overline{1}] : \lambda\!\!\lambda \;,\; 1 \;)$$

T	$\rightarrow$	$(\; T[\epsilon] \;,\; \epsilon \;,\; 0 \;)$
$(\; (n+1)[\mathrm{C} : \rho] \;,\; S \;,\; l \;)$	$\rightarrow$	$(\; n[\rho] \;,\; S \;,\; l \;)$
$(\; 0[\mathrm{C} : \rho] \;,\; S \;,\; l \;)$	$\rightarrow$	$(\; \mathrm{C} \;,\; S \;,\; l \;)$
$(\; (M\,N)[\rho] \;,\; S \;,\; l \;)$	$\rightarrow$	$(\; M[\rho] \;,\; N[\rho] : S \;,\; l \;)$
$(\; (\lambda.B)[\rho] \;,\; N[\rho'] : S \;,\; l \;)$	$\rightarrow$	$(\; B[N[\rho'] : \rho] \;,\; S \;,\; l \;)$
$(\; (\lambda.B)[\rho] \;,\; S \;,\; l \;)$	$\rightarrow$	$(\; B[\overline{l+1} : \rho] \;,\; \lambda\!\!\lambda : S \;,\; l+1 \;)$
$(\; \overline{n} \;,\; S \;,\; l \;)$	$\rightarrow$	$(\; \lfloor l-n, l \rfloor \;,\; S \;,\; l \;)$
$(\; \lfloor M, l' \rfloor \;,\; N[\rho] : S \;,\; l \;)$	$\rightarrow$	$(\; N[\rho] \;,\; \lfloor M, l' \rfloor : S \;,\; l' \;)$
$(\; \lfloor B, l' \rfloor \;,\; \lambda\!\!\lambda : S \;,\; l \;)$	$\rightarrow$	$(\; \lfloor \lambda.B, l' \rfloor \;,\; S \;,\; l \;)$
$(\; \lfloor N, l' \rfloor \;,\; \lfloor M, l'' \rfloor : S \;,\; l \;)$	$\rightarrow$	$(\; \lfloor M\,N, l'' \rfloor \;,\; S \;,\; l \;)$
$(\; \lfloor T, l' \rfloor \;,\; \epsilon \;,\; l \;)$	$\rightarrow$	T

Example

$$(\lfloor 0, 1 \rfloor \ , \ ((\lambda.0)0)[\overline{1}] : \lambda\!\!\lambda \ , \ 1 \ )$$

T	$\rightarrow$	$(\ T[\epsilon] \ , \ \epsilon \ , \ 0 \ )$
$(\ (n+1)[\mathsf{C} : \rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ n[\rho] \ , \ S \ , \ l \ )$
$(\ 0[\mathsf{C} : \rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ \mathsf{C} \ , \ S \ , \ l \ )$
$(\ (M\ N)[\rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ M[\rho] \ , \ N[\rho] : S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ N[\rho'] : S \ , \ l \ )$	$\rightarrow$	$(\ B[N[\rho'] : \rho] \ , \ S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ B[\overline{l+1} : \rho] \ , \ \lambda\!\!\lambda : S \ , \ l+1 \ )$
$(\ \overline{n} \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ \lfloor l-n, l \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor M, l' \rfloor \ , \ N[\rho] : S \ , \ l \ )$	$\rightarrow$	$(\ N[\rho] \ , \ \lfloor M, l' \rfloor : S \ , \ l' \ )$
$(\ \lfloor B, l' \rfloor \ , \ \lambda\!\!\lambda : S \ , \ l \ )$	$\rightarrow$	$(\ \lfloor \lambda.B, l' \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor N, l' \rfloor \ , \ \lfloor M, l'' \rfloor : S \ , \ l \ )$	$\rightarrow$	$(\ \lfloor M\ N, l'' \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor T, l' \rfloor \ , \ \epsilon \ , \ l \ )$	$\rightarrow$	T

Example

$$(\ \lfloor 0,1 \rfloor\ ,\ ((\lambda.0)0)[\overline{1}] : \lambda\!\!\lambda\ ,\ 1\ )$$

T	$\rightarrow$	$(\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ )$
$(\ (n+1)[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ n[\rho]\ ,\ S\ ,\ l\ )$
$(\ 0[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \mathsf{C}\ ,\ S\ ,\ l\ )$
$(\ (M\ N)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ )$	$\rightarrow$	$(\ B[N[\rho'] : \rho]\ ,\ S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ )$
$(\ \overline{n}\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor l-n, l \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor M, l' \rfloor\ ,\ N[\rho] : S\ ,\ l\ )$	$\rightarrow$	$(\ N[\rho]\ ,\ \lfloor M, l' \rfloor : S\ ,\ l'\ )$
$(\ \lfloor B, l' \rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor \lambda.B, l' \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor N, l' \rfloor\ ,\ \lfloor M, l'' \rfloor : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor M\ N, l'' \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor T, l' \rfloor\ ,\ \epsilon\ ,\ l\ )$	$\rightarrow$	T

Example

$$(\ ((\lambda.0)0)[\overline{1}]\ ,\ \lfloor 0,1 \rfloor : \lambda\!\!\lambda\ ,\ 1\ )$$

T	$\rightarrow$	$(\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ )$
$(\ (n+1)[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ n[\rho]\ ,\ S\ ,\ l\ )$
$(\ 0[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \mathsf{C}\ ,\ S\ ,\ l\ )$
$(\ (M\ N)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ )$	$\rightarrow$	$(\ B[N[\rho'] : \rho]\ ,\ S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ )$
$(\ \overline{n}\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor l-n, l \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor M, l' \rfloor\ ,\ N[\rho] : S\ ,\ l\ )$	$\rightarrow$	$(\ N[\rho]\ ,\ \lfloor M, l' \rfloor : S\ ,\ l'\ )$
$(\ \lfloor B, l' \rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor \lambda.B, l' \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor N, l' \rfloor\ ,\ \lfloor M, l'' \rfloor : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor M\ N, l'' \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor T, l' \rfloor\ ,\ \epsilon\ ,\ l\ )$	$\rightarrow$	T

Example

$$(\; ((\lambda.0)0)[\overline{1}] \; , \; \lfloor 0,1 \rfloor : \lambda\!\!\lambda \; , \; 1 \;)$$

T	$\to$	$(\; T[\epsilon] \; , \; \epsilon \; , \; 0 \;)$
$(\; (n+1)[\mathsf{C} : \rho] \; , \; S \; , \; l \;)$	$\to$	$(\; n[\rho] \; , \; S \; , \; l \;)$
$(\; 0[\mathsf{C} : \rho] \; , \; S \; , \; l \;)$	$\to$	$(\; \mathsf{C} \; , \; S \; , \; l \;)$
$(\; (M\,N)[\rho] \; , \; S \; , \; l \;)$	$\to$	$(\; M[\rho] \; , \; N[\rho] : S \; , \; l \;)$
$(\; (\lambda.B)[\rho] \; , \; N[\rho'] : S \; , \; l \;)$	$\to$	$(\; B[N[\rho'] : \rho] \; , \; S \; , \; l \;)$
$(\; (\lambda.B)[\rho] \; , \; S \; , \; l \;)$	$\to$	$(\; B[\overline{l+1} : \rho] \; , \; \lambda\!\!\lambda : S \; , \; l+1 \;)$
$(\; \overline{n} \; , \; S \; , \; l \;)$	$\to$	$(\; \lfloor l-n, l \rfloor \; , \; S \; , \; l \;)$
$(\; \lfloor M, l' \rfloor \; , \; N[\rho] : S \; , \; l \;)$	$\to$	$(\; N[\rho] \; , \; \lfloor M, l' \rfloor : S \; , \; l' \;)$
$(\; \lfloor B, l' \rfloor \; , \; \lambda\!\!\lambda : S \; , \; l \;)$	$\to$	$(\; \lfloor \lambda.B, l' \rfloor \; , \; S \; , \; l \;)$
$(\; \lfloor N, l' \rfloor \; , \; \lfloor M, l'' \rfloor : S \; , \; l \;)$	$\to$	$(\; \lfloor M\,N, l'' \rfloor \; , \; S \; , \; l \;)$
$(\; \lfloor T, l' \rfloor \; , \; \epsilon \; , \; l \;)$	$\to$	T

Example

$$(\ (\lambda.0)[\overline{1}] \ , \ 0[\overline{1}] : \lfloor 0, 1 \rfloor : \lambda\!\!\lambda \ , \ 1 \ )$$

$$
\begin{array}{|rcl|}
\hline
T & \rightarrow & (\ T[\epsilon] \ , \ \epsilon \ , \ 0 \ ) \\
\hline
(\ (n+1)[\mathsf{C} : \rho] \ , \ S \ , \ l \ ) & \rightarrow & (\ n[\rho] \ , \ S \ , \ l \ ) \\
(\ 0[\mathsf{C} : \rho] \ , \ S \ , \ l \ ) & \rightarrow & (\ \mathsf{C} \ , \ S \ , \ l \ ) \\
(\ (M\ N)[\rho] \ , \ S \ , \ l \ ) & \rightarrow & (\ M[\rho] \ , \ N[\rho] : S \ , \ l \ ) \\
(\ (\lambda.B)[\rho] \ , \ N[\rho'] : S \ , \ l \ ) & \rightarrow & (\ B[\overline{N[\rho']} : \rho] \ , \ S \ , \ l \ ) \\
(\ (\lambda.B)[\rho] \ , \ S \ , \ l \ ) & \rightarrow & (\ B[\overline{l+1} : \rho] \ , \ \lambda\!\!\lambda : S \ , \ l+1 \ ) \\
(\ \overline{n} \ , \ S \ , \ l \ ) & \rightarrow & (\ \lfloor l-n, l \rfloor \ , \ S \ , \ l \ ) \\
(\ \lfloor M, l' \rfloor \ , \ N[\rho] : S \ , \ l \ ) & \rightarrow & (\ N[\rho] \ , \ \lfloor M, l' \rfloor : S \ , \ l' \ ) \\
(\ \lfloor B, l' \rfloor \ , \ \lambda\!\!\lambda : S \ , \ l \ ) & \rightarrow & (\ \lfloor \lambda.B, l' \rfloor \ , \ S \ , \ l \ ) \\
(\ \lfloor N, l' \rfloor \ , \ \lfloor M, l'' \rfloor : S \ , \ l \ ) & \rightarrow & (\ \lfloor M\ N, l'' \rfloor \ , \ S \ , \ l \ ) \\
\hline
(\ \lfloor T, l' \rfloor \ , \ \epsilon \ , \ l \ ) & \rightarrow & T \\
\hline
\end{array}
$$

Example

$$(\ (\lambda.0)[\overline{1}]\ ,\ 0[\overline{1}] : \lfloor 0,1 \rfloor : \lambda\!\!\lambda\ ,\ 1\ )$$

T	$\rightarrow$	$(\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ )$
$(\ (n+1)[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ n[\rho]\ ,\ S\ ,\ l\ )$
$(\ 0[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \mathsf{C}\ ,\ S\ ,\ l\ )$
$(\ (M\ N)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ )$	$\rightarrow$	$(\ B[N[\rho'] : \rho]\ ,\ S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ )$
$(\ \overline{n}\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor l-n, l \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor M, l' \rfloor\ ,\ N[\rho] : S\ ,\ l\ )$	$\rightarrow$	$(\ N[\rho]\ ,\ \lfloor M, l' \rfloor : S\ ,\ l'\ )$
$(\ \lfloor B, l' \rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor \lambda.B, l' \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor N, l' \rfloor\ ,\ \lfloor M, l'' \rfloor : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor M\ N, l'' \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor T, l' \rfloor\ ,\ \epsilon\ ,\ l\ )$	$\rightarrow$	T

Example

$$(\ 0[0[\overline{1}] : \overline{1}]\ ,\ \lfloor 0, 1 \rfloor : \lambda\!\!\lambda\ ,\ 1\ )$$

T	$\rightarrow$	$(\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ )$
$(\ (n+1)[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ n[\rho]\ ,\ S\ ,\ l\ )$
$(\ 0[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \mathsf{C}\ ,\ S\ ,\ l\ )$
$(\ (M\ N)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ )$	$\rightarrow$	$(\ B[N[\rho'] : \rho]\ ,\ S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ )$
$(\ \overline{n}\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor l-n, l \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor M, l' \rfloor\ ,\ N[\rho] : S\ ,\ l\ )$	$\rightarrow$	$(\ N[\rho]\ ,\ \lfloor M, l' \rfloor : S\ ,\ l'\ )$
$(\ \lfloor B, l' \rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor \lambda.B, l' \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor N, l' \rfloor\ ,\ \lfloor M, l'' \rfloor : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor M\ N, l'' \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor T, l' \rfloor\ ,\ \epsilon\ ,\ l\ )$	$\rightarrow$	T

Example

$$(\ 0[0[\overline{1}] : \overline{1}] \ , \ \lfloor 0, 1 \rfloor : \lambda\!\!\!\lambda \ , \ 1 \ )$$

$$
\begin{array}{|rcl|}
\hline
T & \to & (\ T[\epsilon] \ , \ \epsilon \ , \ 0 \ ) \\
\hline
(\ (n+1)[\mathrm{C} : \rho] \ , \ S \ , \ l \ ) & \to & (\ n[\rho] \ , \ S \ , \ l \ ) \\
(\ 0[\mathrm{C} : \rho] \ , \ S \ , \ l \ ) & \to & (\ \mathrm{C} \ , \ S \ , \ l \ ) \\
(\ (M\ N)[\rho] \ , \ S \ , \ l \ ) & \to & (\ M[\rho] \ , \ N[\rho] : S \ , \ l \ ) \\
(\ (\lambda.B)[\rho] \ , \ N[\rho'] : S \ , \ l \ ) & \to & (\ B[\overline{N[\rho']} : \rho] \ , \ S \ , \ l \ ) \\
(\ (\lambda.B)[\rho] \ , \ S \ , \ l \ ) & \to & (\ B[\overline{l+1} : \rho] \ , \ \lambda\!\!\!\lambda : S \ , \ l+1 \ ) \\
(\ \overline{n} \ , \ S \ , \ l \ ) & \to & (\ \lfloor l-n, l \rfloor \ , \ S \ , \ l \ ) \\
(\ \lfloor M, l' \rfloor \ , \ N[\rho] : S \ , \ l \ ) & \to & (\ N[\rho] \ , \ \lfloor M, l' \rfloor : S \ , \ l' \ ) \\
(\ \lfloor B, l' \rfloor \ , \ \lambda\!\!\!\lambda : S \ , \ l \ ) & \to & (\ \lfloor \lambda.B, l' \rfloor \ , \ S \ , \ l \ ) \\
(\ \lfloor N, l' \rfloor \ , \ \lfloor M, l'' \rfloor : S \ , \ l \ ) & \to & (\ \lfloor M\ N, l'' \rfloor \ , \ S \ , \ l \ ) \\
\hline
(\ \lfloor T, l' \rfloor \ , \ \epsilon \ , \ l \ ) & \to & T \\
\hline
\end{array}
$$

Example

$$(\; 0[\overline{1}] \; , \; \lfloor 0, 1 \rfloor : \lambda\!\!\lambda \; , \; 1 \;)$$

T	$\rightarrow$	$(\; T[\epsilon] \; , \; \epsilon \; , \; 0 \;)$
$(\; (n+1)[\mathrm{C} : \rho] \; , \; S \; , \; l \;)$	$\rightarrow$	$(\; n[\rho] \; , \; S \; , \; l \;)$
$(\; 0[\mathrm{C} : \rho] \; , \; S \; , \; l \;)$	$\rightarrow$	$(\; \mathrm{C} \; , \; S \; , \; l \;)$
$(\; (M\,N)[\rho] \; , \; S \; , \; l \;)$	$\rightarrow$	$(\; M[\rho] \; , \; N[\rho] : S \; , \; l \;)$
$(\; (\lambda.B)[\rho] \; , \; N[\rho'] : S \; , \; l \;)$	$\rightarrow$	$(\; B[N[\rho'] : \rho] \; , \; S \; , \; l \;)$
$(\; (\lambda.B)[\rho] \; , \; S \; , \; l \;)$	$\rightarrow$	$(\; B[\overline{l+1} : \rho] \; , \; \lambda\!\!\lambda : S \; , \; l+1 \;)$
$(\; \overline{n} \; , \; S \; , \; l \;)$	$\rightarrow$	$(\; \lfloor l-n, l \rfloor \; , \; S \; , \; l \;)$
$(\; \lfloor M, l' \rfloor \; , \; N[\rho] : S \; , \; l \;)$	$\rightarrow$	$(\; N[\rho] \; , \; \lfloor M, l' \rfloor : S \; , \; l' \;)$
$(\; \lfloor B, l' \rfloor \; , \; \lambda\!\!\lambda : S \; , \; l \;)$	$\rightarrow$	$(\; \lfloor \lambda.B, l' \rfloor \; , \; S \; , \; l \;)$
$(\; \lfloor N, l' \rfloor \; , \; \lfloor M, l'' \rfloor : S \; , \; l \;)$	$\rightarrow$	$(\; \lfloor M\,N, l'' \rfloor \; , \; S \; , \; l \;)$
$(\; \lfloor T, l' \rfloor \; , \; \epsilon \; , \; l \;)$	$\rightarrow$	T

Example

$$(\ 0[\overline{1}] \ , \ \lfloor 0, 1 \rfloor : \lambda\!\!\lambda \ , \ 1 \ )$$

T	$\rightarrow$	$(\ T[\epsilon] \ , \ \epsilon \ , \ 0 \ )$
$(\ (n+1)[\mathsf{C} : \rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ n[\rho] \ , \ S \ , \ l \ )$
$(\ 0[\mathsf{C} : \rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ \mathsf{C} \ , \ S \ , \ l \ )$
$(\ (M\ N)[\rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ M[\rho] \ , \ N[\rho] : S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ N[\rho'] : S \ , \ l \ )$	$\rightarrow$	$(\ B[N[\rho'] : \rho] \ , \ S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ B[\overline{l+1} : \rho] \ , \ \lambda\!\!\lambda : S \ , \ l+1 \ )$
$(\ \overline{n} \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ \lfloor l-n, l \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor M, l' \rfloor \ , \ N[\rho] : S \ , \ l \ )$	$\rightarrow$	$(\ N[\rho] \ , \ \lfloor M, l' \rfloor : S \ , \ l' \ )$
$(\ \lfloor B, l' \rfloor \ , \ \lambda\!\!\lambda : S \ , \ l \ )$	$\rightarrow$	$(\ \lfloor \lambda.B, l' \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor N, l' \rfloor \ , \ \lfloor M, l'' \rfloor : S \ , \ l \ )$	$\rightarrow$	$(\ \lfloor M\ N, l'' \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor T, l' \rfloor \ , \ \epsilon \ , \ l \ )$	$\rightarrow$	T

Example

$$(\overline{1} \ , \ \lfloor 0, 1 \rfloor : \lambda\!\!\lambda \ , \ 1 \ )$$

T	$\to$	$(\ T[\epsilon] \ , \ \epsilon \ , \ 0 \ )$
$(\ (n+1)[\mathsf{C} : \rho] \ , \ S \ , \ l \ )$	$\to$	$(\ n[\rho] \ , \ S \ , \ l \ )$
$(\ 0[\mathsf{C} : \rho] \ , \ S \ , \ l \ )$	$\to$	$(\ \mathsf{C} \ , \ S \ , \ l \ )$
$(\ (M\ N)[\rho] \ , \ S \ , \ l \ )$	$\to$	$(\ M[\rho] \ , \ N[\rho] : S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ N[\rho'] : S \ , \ l \ )$	$\to$	$(\ B[\overline{N[\rho']} : \rho] \ , \ S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ S \ , \ l \ )$	$\to$	$(\ B[\overline{l+1} : \rho] \ , \ \lambda\!\!\lambda : S \ , \ l+1 \ )$
$(\ \overline{n} \ , \ S \ , \ l \ )$	$\to$	$(\ \lfloor l - n, l \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor M, l' \rfloor \ , \ N[\rho] : S \ , \ l \ )$	$\to$	$(\ N[\rho] \ , \ \lfloor M, l' \rfloor : S \ , \ l' \ )$
$(\ \lfloor B, l' \rfloor \ , \ \lambda\!\!\lambda : S \ , \ l \ )$	$\to$	$(\ \lfloor \lambda.B, l' \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor N, l' \rfloor \ , \ \lfloor M, l'' \rfloor : S \ , \ l \ )$	$\to$	$(\ \lfloor M\ N, l'' \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor T, l' \rfloor \ , \ \epsilon \ , \ l \ )$	$\to$	T

Example

$$(\; \overline{1} \; , \; \lfloor 0,1 \rfloor : \lambda\!\!\lambda \; , \; 1 \;)$$

T	$\rightarrow$	$(\; T[\epsilon] \; , \; \epsilon \; , \; 0 \;)$
$(\; (n+1)[\mathsf{C} : \rho] \; , \; S \; , \; l \;)$	$\rightarrow$	$(\; n[\rho] \; , \; S \; , \; l \;)$
$(\; 0[\mathsf{C} : \rho] \; , \; S \; , \; l \;)$	$\rightarrow$	$(\; \mathsf{C} \; , \; S \; , \; l \;)$
$(\; (M\;N)[\rho] \; , \; S \; , \; l \;)$	$\rightarrow$	$(\; M[\rho] \; , \; N[\rho] : S \; , \; l \;)$
$(\; (\lambda.B)[\rho] \; , \; N[\rho'] : S \; , \; l \;)$	$\rightarrow$	$(\; B[N[\rho'] : \rho] \; , \; S \; , \; l \;)$
$(\; (\lambda.B)[\rho] \; , \; S \; , \; l \;)$	$\rightarrow$	$(\; B[\overline{l+1} : \rho] \; , \; \lambda\!\!\lambda : S \; , \; l+1 \;)$
$(\; \overline{n} \; , \; S \; , \; l \;)$	$\rightarrow$	$(\; \lfloor l-n, l \rfloor \; , \; S \; , \; l \;)$
$(\; \lfloor M, l' \rfloor \; , \; N[\rho] : S \; , \; l \;)$	$\rightarrow$	$(\; N[\rho] \; , \; \lfloor M, l' \rfloor : S \; , \; l' \;)$
$(\; \lfloor B, l' \rfloor \; , \; \lambda\!\!\lambda : S \; , \; l \;)$	$\rightarrow$	$(\; \lfloor \lambda.B, l' \rfloor \; , \; S \; , \; l \;)$
$(\; \lfloor N, l' \rfloor \; , \; \lfloor M, l'' \rfloor : S \; , \; l \;)$	$\rightarrow$	$(\; \lfloor M\;N, l'' \rfloor \; , \; S \; , \; l \;)$
$(\; \lfloor T, l' \rfloor \; , \; \epsilon \; , \; l \;)$	$\rightarrow$	T

Example

$$(\lfloor 0, 1 \rfloor \ , \ \lfloor 0, 1 \rfloor : \lambda\!\!\lambda \ , \ 1 \ )$$

T	$\rightarrow$	$(\ T[\epsilon] \ , \ \epsilon \ , \ 0 \ )$
$(\ (n+1)[\mathsf{C} : \rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ n[\rho] \ , \ S \ , \ l \ )$
$(\ 0[\mathsf{C} : \rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ \mathsf{C} \ , \ S \ , \ l \ )$
$(\ (M\ N)[\rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ M[\rho] \ , \ N[\rho] : S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ N[\rho'] : S \ , \ l \ )$	$\rightarrow$	$(\ B[N[\rho'] : \rho] \ , \ S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ B[\overline{l+1} : \rho] \ , \ \lambda\!\!\lambda : S \ , \ l+1 \ )$
$(\ \overline{n} \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ \lfloor l-n, l \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor M, l' \rfloor \ , \ N[\rho] : S \ , \ l \ )$	$\rightarrow$	$(\ N[\rho] \ , \ \lfloor M, l' \rfloor : S \ , \ l' \ )$
$(\ \lfloor B, l' \rfloor \ , \ \lambda\!\!\lambda : S \ , \ l \ )$	$\rightarrow$	$(\ \lfloor \lambda.B, l' \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor N, l' \rfloor \ , \ \lfloor M, l'' \rfloor : S \ , \ l \ )$	$\rightarrow$	$(\ \lfloor M\ N, l'' \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor T, l' \rfloor \ , \ \epsilon \ , \ l \ )$	$\rightarrow$	T

Example

$$(\lfloor 0, 1 \rfloor \; , \; \lfloor 0, 1 \rfloor : \lambda\!\!\lambda \; , \; 1)$$

T	$\rightarrow$	$(\; T[\epsilon] \; , \; \epsilon \; , \; 0 \;)$
$(\; (n+1)[\mathsf{C} : \rho] \; , \; S \; , \; l \;)$	$\rightarrow$	$(\; n[\rho] \; , \; S \; , \; l \;)$
$(\; 0[\mathsf{C} : \rho] \; , \; S \; , \; l \;)$	$\rightarrow$	$(\; \mathsf{C} \; , \; S \; , \; l \;)$
$(\; (M \; N)[\rho] \; , \; S \; , \; l \;)$	$\rightarrow$	$(\; M[\rho] \; , \; N[\rho] : S \; , \; l \;)$
$(\; (\lambda.B)[\rho] \; , \; N[\rho'] : S \; , \; l \;)$	$\rightarrow$	$(\; B[N[\rho'] : \rho] \; , \; S \; , \; l \;)$
$(\; (\lambda.B)[\rho] \; , \; S \; , \; l \;)$	$\rightarrow$	$(\; B[\overline{l+1} : \rho] \; , \; \lambda\!\!\lambda : S \; , \; l+1 \;)$
$(\; \overline{n} \; , \; S \; , \; l \;)$	$\rightarrow$	$(\; \lfloor l - n, l \rfloor \; , \; S \; , \; l \;)$
$(\; \lfloor M, l' \rfloor \; , \; N[\rho] : S \; , \; l \;)$	$\rightarrow$	$(\; N[\rho] \; , \; \lfloor M, l' \rfloor : S \; , \; l' \;)$
$(\; \lfloor B, l' \rfloor \; , \; \lambda\!\!\lambda : S \; , \; l \;)$	$\rightarrow$	$(\; \lfloor \lambda.B, l' \rfloor \; , \; S \; , \; l \;)$
$(\; \lfloor N, l' \rfloor \; , \; \lfloor M, l'' \rfloor : S \; , \; l \;)$	$\rightarrow$	$(\; \lfloor M \; N, l'' \rfloor \; , \; S \; , \; l \;)$
$(\; \lfloor T, l' \rfloor \; , \; \epsilon \; , \; l \;)$	$\rightarrow$	T

Example

$$(\lfloor 0\;0, 1 \rfloor \;,\; \lambda\!\!\lambda \;,\; 1 \;)$$

T	$\rightarrow$	$(\; T[\epsilon] \;,\; \epsilon \;,\; 0 \;)$
$(\; (n+1)[\mathrm{C} : \rho] \;,\; S \;,\; l \;)$	$\rightarrow$	$(\; n[\rho] \;,\; S \;,\; l \;)$
$(\; 0[\mathrm{C} : \rho] \;,\; S \;,\; l \;)$	$\rightarrow$	$(\; \mathrm{C} \;,\; S \;,\; l \;)$
$(\; (M\;N)[\rho] \;,\; S \;,\; l \;)$	$\rightarrow$	$(\; M[\rho] \;,\; N[\rho] : S \;,\; l \;)$
$(\; (\lambda.B)[\rho] \;,\; N[\rho'] : S \;,\; l \;)$	$\rightarrow$	$(\; B[N[\rho'] : \rho] \;,\; S \;,\; l \;)$
$(\; (\lambda.B)[\rho] \;,\; S \;,\; l \;)$	$\rightarrow$	$(\; B[\overline{l+1} : \rho] \;,\; \lambda\!\!\lambda : S \;,\; l+1 \;)$
$(\; \overline{n} \;,\; S \;,\; l \;)$	$\rightarrow$	$(\; \lfloor l-n, l \rfloor \;,\; S \;,\; l \;)$
$(\; \lfloor M, l' \rfloor \;,\; N[\rho] : S \;,\; l \;)$	$\rightarrow$	$(\; N[\rho] \;,\; \lfloor M, l' \rfloor : S \;,\; l' \;)$
$(\; \lfloor B, l' \rfloor \;,\; \lambda\!\!\lambda : S \;,\; l \;)$	$\rightarrow$	$(\; \lfloor \lambda.B, l' \rfloor \;,\; S \;,\; l \;)$
$(\; \lfloor N, l' \rfloor \;,\; \lfloor M, l'' \rfloor : S \;,\; l \;)$	$\rightarrow$	$(\; \lfloor M\;N, l'' \rfloor \;,\; S \;,\; l \;)$
$(\; \lfloor T, l' \rfloor \;,\; \epsilon \;,\; l \;)$	$\rightarrow$	T

Example

$$(\lfloor 0\,0, 1 \rfloor \; , \; \lambda\!\!\lambda \; , \; 1)$$

T	$\rightarrow$	$(T[\epsilon] \; , \; \epsilon \; , \; 0)$
$((n+1)[\mathsf{C} : \rho] \; , \; S \; , \; l)$	$\rightarrow$	$(n[\rho] \; , \; S \; , \; l)$
$(0[\mathsf{C} : \rho] \; , \; S \; , \; l)$	$\rightarrow$	$(\mathsf{C} \; , \; S \; , \; l)$
$((M\,N)[\rho] \; , \; S \; , \; l)$	$\rightarrow$	$(M[\rho] \; , \; N[\rho] : S \; , \; l)$
$((\lambda.B)[\rho] \; , \; N[\rho'] : S \; , \; l)$	$\rightarrow$	$(B[N[\rho'] : \rho] \; , \; S \; , \; l)$
$((\lambda.B)[\rho] \; , \; S \; , \; l)$	$\rightarrow$	$(B[\overline{l+1} : \rho] \; , \; \lambda\!\!\lambda : S \; , \; l+1)$
$(\overline{n} \; , \; S \; , \; l)$	$\rightarrow$	$(\lfloor l-n, l \rfloor \; , \; S \; , \; l)$
$(\lfloor M, l' \rfloor \; , \; N[\rho] : S \; , \; l)$	$\rightarrow$	$(N[\rho] \; , \; \lfloor M, l' \rfloor : S \; , \; l')$
$(\lfloor B, l' \rfloor \; , \; \lambda\!\!\lambda : S \; , \; l)$	$\rightarrow$	$(\lfloor \lambda.B, l' \rfloor \; , \; S \; , \; l)$
$(\lfloor N, l' \rfloor \; , \; \lfloor M, l'' \rfloor : S \; , \; l)$	$\rightarrow$	$(\lfloor M\,N, l'' \rfloor \; , \; S \; , \; l)$
$(\lfloor T, l' \rfloor \; , \; \epsilon \; , \; l)$	$\rightarrow$	T

Example

$$(\lfloor \lambda.0\,0, 1 \rfloor \ , \ \epsilon \ , \ 1 \ )$$

T	$\to$	$(\ T[\epsilon] \ , \ \epsilon \ , \ 0 \ )$
$(\ (n+1)[\mathsf{C} : \rho] \ , \ S \ , \ l \ )$	$\to$	$(\ n[\rho] \ , \ S \ , \ l \ )$
$(\ 0[\mathsf{C} : \rho] \ , \ S \ , \ l \ )$	$\to$	$(\ \mathsf{C} \ , \ S \ , \ l \ )$
$(\ (M\,N)[\rho] \ , \ S \ , \ l \ )$	$\to$	$(\ M[\rho] \ , \ N[\rho] : S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ N[\rho'] : S \ , \ l \ )$	$\to$	$(\ B[\overline{N[\rho']} : \rho] \ , \ S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ S \ , \ l \ )$	$\to$	$(\ B[\overline{l+1} : \rho] \ , \ \lambda\!\!\lambda : S \ , \ l+1 \ )$
$(\ \overline{n} \ , \ S \ , \ l \ )$	$\to$	$(\ \lfloor l-n, l \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor M, l' \rfloor \ , \ N[\rho] : S \ , \ l \ )$	$\to$	$(\ N[\rho] \ , \ \lfloor M, l' \rfloor : S \ , \ l' \ )$
$(\ \lfloor B, l' \rfloor \ , \ \lambda\!\!\lambda : S \ , \ l \ )$	$\to$	$(\ \lfloor \lambda.B, l' \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor N, l' \rfloor \ , \ \lfloor M, l'' \rfloor : S \ , \ l \ )$	$\to$	$(\ \lfloor M\,N, l'' \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor T, l' \rfloor \ , \ \epsilon \ , \ l \ )$	$\to$	T

Example

$$(\lfloor \lambda.0\,0, 1 \rfloor \;,\; \epsilon \;,\; 1)$$

T	$\rightarrow$	$(T[\epsilon] \;,\; \epsilon \;,\; 0)$
$((n+1)[\mathsf{C} : \rho] \;,\; S \;,\; l)$	$\rightarrow$	$(n[\rho] \;,\; S \;,\; l)$
$(0[\mathsf{C} : \rho] \;,\; S \;,\; l)$	$\rightarrow$	$(\mathsf{C} \;,\; S \;,\; l)$
$((M\,N)[\rho] \;,\; S \;,\; l)$	$\rightarrow$	$(M[\rho] \;,\; N[\rho] : S \;,\; l)$
$((\lambda.B)[\rho] \;,\; N[\rho'] : S \;,\; l)$	$\rightarrow$	$(B[N[\rho'] : \rho] \;,\; S \;,\; l)$
$((\lambda.B)[\rho] \;,\; S \;,\; l)$	$\rightarrow$	$(B[\overline{l+1} : \rho] \;,\; \lambda\!\!\!\lambda : S \;,\; l+1)$
$(\overline{n} \;,\; S \;,\; l)$	$\rightarrow$	$(\lfloor l-n, l \rfloor \;,\; S \;,\; l)$
$(\lfloor M, l' \rfloor \;,\; N[\rho] : S \;,\; l)$	$\rightarrow$	$(N[\rho] \;,\; \lfloor M, l' \rfloor : S \;,\; l')$
$(\lfloor B, l' \rfloor \;,\; \lambda\!\!\!\lambda : S \;,\; l)$	$\rightarrow$	$(\lfloor \lambda.B, l' \rfloor \;,\; S \;,\; l)$
$(\lfloor N, l' \rfloor \;,\; \lfloor M, l'' \rfloor : S \;,\; l)$	$\rightarrow$	$(\lfloor M\,N, l'' \rfloor \;,\; S \;,\; l)$
$(\lfloor T, l' \rfloor \;,\; \epsilon \;,\; l)$	$\rightarrow$	T

Example

$$\lambda.0\,0$$

$$
\begin{array}{|rcl|}
\hline
T & \to & (\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ ) \\
\hline
(\ (n+1)[\mathrm{C} : \rho]\ ,\ S\ ,\ l\ ) & \to & (\ n[\rho]\ ,\ S\ ,\ l\ ) \\
(\ 0[\mathrm{C} : \rho]\ ,\ S\ ,\ l\ ) & \to & (\ \mathrm{C}\ ,\ S\ ,\ l\ ) \\
(\ (M\ N)[\rho]\ ,\ S\ ,\ l\ ) & \to & (\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ ) \\
(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ ) & \to & (\ B[N[\rho'] : \rho]\ ,\ S\ ,\ l\ ) \\
(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ ) & \to & (\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ ) \\
(\ \overline{n}\ ,\ S\ ,\ l\ ) & \to & (\ \lfloor l-n, l \rfloor\ ,\ S\ ,\ l\ ) \\
(\ \lfloor M, l' \rfloor\ ,\ N[\rho] : S\ ,\ l\ ) & \to & (\ N[\rho]\ ,\ \lfloor M, l' \rfloor : S\ ,\ l'\ ) \\
(\ \lfloor B, l' \rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ ) & \to & (\ \lfloor \lambda.B, l' \rfloor\ ,\ S\ ,\ l\ ) \\
(\ \lfloor N, l' \rfloor\ ,\ \lfloor M, l'' \rfloor : S\ ,\ l\ ) & \to & (\ \lfloor M\ N, l'' \rfloor\ ,\ S\ ,\ l\ ) \\
\hline
(\ \lfloor T, l' \rfloor\ ,\ \epsilon\ ,\ l\ ) & \to & T \\
\hline
\end{array}
$$

Calculus of Closures ($\lambda_{\widetilde{\rho}}$)

$$
\begin{array}{lcl}
\mathsf{C} & ::= & \Lambda[\rho] \mid \overline{n} \mid \lfloor n \rfloor \mid \lambda\!\!\lambda.\mathsf{C} \mid \mathsf{C} \cdot \mathsf{C} \\
\rho & ::= & \epsilon \mid \mathsf{C} : \rho
\end{array}
$$

Calculus of Closures ($\lambda_{\widetilde{\rho}}$)

[Curien, 1991]

$$
\begin{array}{lcl}
\mathsf{C} & ::= & \Lambda[\rho] \mid \overline{n} \mid \lfloor n \rfloor \mid \lambda\!\!\lambda.\mathsf{C} \mid \mathsf{C} \cdot \mathsf{C} \\
\rho & ::= & \epsilon \mid \mathsf{C} : \rho
\end{array}
$$

Calculus of Closures ($\lambda_{\widetilde{\rho}}$)

[Biernacka and Danvy, 2007]

$$\begin{array}{rcl} \mathsf{C} & ::= & \Lambda[\rho] \mid \overline{n} \mid \lfloor n \rfloor \mid \lambda\!\!\lambda.\mathsf{C} \mid \mathsf{C} \cdot \mathsf{C} \\ \rho & ::= & \epsilon \mid \mathsf{C} : \rho \end{array}$$

Calculus of Closures ($\lambda_{\widetilde{\rho}}$)

$$
\begin{array}{lcl}
\mathsf{C} & ::= & \Lambda[\rho] \mid \overline{n} \mid \lfloor n \rfloor \mid \lambda\!\!\lambda.\mathsf{C} \mid \mathsf{C} \cdot \mathsf{C} \\
\rho & ::= & \epsilon \mid \mathsf{C} : \rho
\end{array}
$$

Calculus of Closures ($\lambda_{\widetilde{\rho}}$)

$$
\begin{array}{lcl}
\mathsf{C} & ::= & \Lambda[\rho] \mid \overline{n} \mid \lfloor n \rfloor \mid \lambda\!\!\lambda.\mathsf{C} \mid \mathsf{C} \cdot \mathsf{C} \\
\rho & ::= & \epsilon \mid \mathsf{C} : \rho
\end{array}
$$

$$
\mathsf{E} \quad ::= \quad \lfloor n \rfloor \mid \lambda\!\!\lambda.\mathsf{E} \mid \mathsf{E} \cdot \mathsf{E}
$$

Calculus of Closures ($\lambda_{\widetilde{\rho}}$)

$$\begin{array}{lcl} \mathsf{C} & ::= & \Lambda[\rho] \mid \overline{n} \mid \lfloor n \rfloor \mid \lambda\!\!\lambda.\mathsf{C} \mid \mathsf{C} \cdot \mathsf{C} \\ \rho & ::= & \epsilon \mid \mathsf{C} : \rho \end{array}$$

$$\mathsf{E} \quad ::= \quad \lfloor n \rfloor \mid \lambda\!\!\lambda.\mathsf{E} \mid \mathsf{E} \cdot \mathsf{E} \quad (\simeq \ \Lambda)$$

Example

$$\lambda.0((\lambda.0)0)$$

Example

$$\left[\ (\lambda.0((\lambda.0)0))[\epsilon]\ \right]$$

Example

$$\left[\quad \cancel{\lambda}.(0((\lambda.0)0)[\epsilon]) \quad\right]$$

Example

$$\left[\quad \lambda\!\!\lambda.(0((\lambda.0)0)[\epsilon]) \quad \right]_0$$

Example

$$\left[\ \cancel{\lambda}.(0((\lambda.0)0)[\epsilon])\ \right]_0$$

Example

$$\left[\ \lambda\!\!\lambda.(0((\lambda.0)0)[\overline{1}])\ \right]_0$$

Example

$$\lambda\!\!\lambda.\left(\quad \left[\; (0(\lambda.0)0)[\overline{1}] \; \right]_1 \quad \right)$$

Example

$$\lambda\!\!\lambda.\left(\quad \left[\; 0[\bar{1}] \cdot ((\lambda.0)0)[\bar{1}] \; \right]_1 \quad \right)$$

Example

$$\lambda.(\quad \left[\; 0[\overline{1}] \;\right]_1 \quad \cdot ((\lambda.0)0)[\overline{1}])$$

Example

$$\not\lambda.\left(\left[\,0[\bar{1}]\,\right]_1 \cdot ((\lambda.0)0)[\bar{1}]\right)$$

Example

$$\lambda\!\!\lambda.\left(\quad \left[\; \bar{1} \; \right]_1 \quad \cdot ((\lambda.0)0)[\bar{1}] \right)$$

Example

$$\lambda\!\!\lambda.\left(\left[\,\lfloor 1 - \overline{1} \rfloor\,\right]_1 \cdot ((\lambda.0)0)[\overline{1}]\right)$$

Example

$$\cancel{\lambda}.\left(\left[\lfloor 0 \rfloor \right]_1 \cdot ((\lambda.0)0)[\overline{1}] \right)$$

Example

$$\cancel{\lambda}.(\lfloor 0 \rfloor \cdot \quad \left[\; ((\lambda.0)0)[\overline{1}] \; \right]_1 \quad)$$

Example

$$\lambda\!\!\!\lambda.(\lfloor 0 \rfloor \cdot \quad \Big[\; (\lambda.0)[\overline{1}] \cdot 0[\overline{1}] \; \Big]_1 \quad)$$

Example

$$\lambda\!\!\lambda.(\lfloor 0 \rfloor \cdot \left[\; (\lambda.0)[\overline{1}] \; \right]_1 \cdot 0[\overline{1}])$$

Example

$$\lambda\!\!\!\lambda.(\lfloor 0 \rfloor \cdot \left[\lambda\!\!\!\lambda.(0[\overline{1}]) \right]_1 \cdot 0[\overline{1}])$$

Example

$$\lambda\!\!\lambda.(\lfloor 0 \rfloor \cdot \left[\ \lambda\!\!\lambda.(0[\overline{1}])\ \right]_1 \cdot 0[\overline{1}])$$

Example

$$\lambda\!\!\!\lambda.(\lfloor 0 \rfloor \cdot \quad \left[\; \lambda\!\!\!\lambda.(0[\overline{2} : \overline{1}]) \; \right]_1 \quad \cdot\, 0[\overline{1}])$$

Example

$$\lambda\!\!\lambda.(\lfloor 0 \rfloor \cdot \quad \left[\; \lambda\!\!\lambda.(0[\overline{2} : \overline{1}]) \cdot 0[\overline{1}] \; \right]_1 \quad)$$

Example

$$\lambda\!\!\lambda.(\lfloor 0 \rfloor \cdot \quad \left[\; \lambda\!\!\lambda.(0[\overline{2} : \overline{1}]) \cdot 0[\overline{1}] \; \right]_1 \quad)$$

Example

$$\lambda\!\!\lambda.(\lfloor 0 \rfloor \cdot \quad \left[\; 0[0[\overline{1}] : \overline{1}] \; \right]_1 \quad)$$

Example

$$\lambda.\left(\lfloor 0 \rfloor \cdot \quad \left[\; 0[0[\overline{1}] : \overline{1}] \;\right]_1 \quad \right)$$

Example

$$\lambda\!\!\!\lambda.(\lfloor 0 \rfloor \cdot \quad \left[\; 0[\bar{1}] \; \right]_1 \quad)$$

Example

$$\lambda\!\!\!\lambda.(\lfloor 0 \rfloor \cdot \left[\; 0[\bar{1}] \;\right]_1 \;)$$

Example

$$\lambda\!\!\lambda.(\lfloor 0 \rfloor \cdot \quad \left[\ \overline{1}\ \right]_1 \quad)$$

Example

$$\lambda\!\!\lambda.(\lfloor 0 \rfloor \cdot \quad \Big[\; \lfloor 1 - \overline{1} \rfloor \; \Big]_1 \quad)$$

Example

$$\lambda\!\!\lambda.(\lfloor 0 \rfloor \cdot \quad \Big[\ \lfloor 0 \rfloor\ \Big]_1 \quad)$$

Example

$$ƛ.(\lfloor 0 \rfloor \cdot \lfloor 0 \rfloor)$$

Example

$$\lambda.0\,0$$

Structural Operational Semantics ($\lambda_{\widetilde{\rho}}$)

$$\frac{n < |\rho|}{\langle n[\rho], l\rangle \to \langle n^{\text{th}}(\rho), l\rangle}\ (\text{Var}) \qquad \frac{}{\langle(\lambda\!\!\lambda.B[\overline{n}:\rho])\cdot \mathsf{N}, l\rangle \to \langle B[\mathsf{N}:\rho], l\rangle}\ (\beta)$$

$$\frac{}{\langle(M\ N)[\rho], l\rangle \to \langle M[\rho]\cdot N[\rho], l\rangle}\ (\text{App}) \qquad \frac{\mathsf{M} \notin \mathrm{WHNF}_{\mathsf{C}} \quad \langle \mathsf{M}, l\rangle \to \langle \mathsf{M}', l\rangle}{\langle \mathsf{M}\cdot\mathsf{N}, l\rangle \to \langle \mathsf{M}'\cdot\mathsf{N}, l\rangle}\ (\mu 1)$$

$$\frac{}{\langle \overline{n}, l\rangle \to \langle \lfloor l-n\rfloor, l\rangle}\ (\text{Par}) \qquad \frac{\mathsf{M} \in \mathrm{WHNF}_{\mathsf{C}} \quad \mathsf{M} \not\equiv \lambda\!\!\lambda.\mathsf{B} \quad \langle \mathsf{M}, l\rangle \to \langle \mathsf{M}', l\rangle}{\langle \mathsf{M}\cdot\mathsf{N}, l\rangle \to \langle \mathsf{M}'\cdot\mathsf{N}, l\rangle}\ (\mu 2)$$

$$\frac{n \geq |\rho|}{\langle n[\rho], l\rangle \to \langle \lfloor n-(|\rho|-l)\rfloor, l\rangle}\ (\text{Fre}) \qquad \frac{\mathsf{M} \in \mathrm{NF}_{\mathsf{C}} \quad \mathsf{M} \not\equiv \lambda\!\!\lambda.\mathsf{B} \quad \langle \mathsf{N}, l\rangle \to \langle \mathsf{N}', l\rangle}{\langle \mathsf{M}\cdot\mathsf{N}, l\rangle \to \langle \mathsf{M}\cdot\mathsf{N}', l\rangle}\ (\nu)$$

$$\frac{}{\langle(\lambda.B)[\rho], l\rangle \to \langle \lambda\!\!\lambda.B[\overline{l+1}:\rho], l\rangle}\ (\text{Lam}) \qquad \frac{\langle \mathsf{B}, l+1\rangle \to \langle \mathsf{B}', l+1\rangle}{\langle \lambda\!\!\lambda.\mathsf{B}, l\rangle \to \langle \lambda\!\!\lambda.\mathsf{B}', l\rangle}\ (\xi)$$

$$\begin{array}{rcl} \mathrm{WHNF}_{\mathsf{C}} & ::= & \lambda\!\!\lambda.\mathsf{C} \mid \lfloor n\rfloor\{\cdot\mathsf{C}\}^* \\ \mathrm{NF}_{\mathsf{C}} & ::= & \lambda\!\!\lambda.\mathrm{NF}_{\mathsf{C}} \mid \lfloor n\rfloor\{\cdot\mathrm{NF}_{\mathsf{C}}\}^* \end{array}$$

Structural Operational Semantics ($\lambda_{\widetilde{\rho}}$)

$$\frac{n < |\rho|}{\langle n[\rho], l\rangle \to \langle n^{\mathrm{th}}(\rho), l\rangle}\ (\mathsf{Var}) \qquad \frac{}{\langle (\lambda\!\!\lambda.B[\overline{n} : \rho]) \cdot \mathsf{N}, l\rangle \to \langle B[\mathsf{N} : \rho], l\rangle}\ (\beta)$$

$$\frac{}{\langle (M\ N)[\rho], l\rangle \to \langle M[\rho] \cdot N[\rho], l\rangle}\ (\mathsf{App}) \qquad \frac{\mathsf{M} \notin \mathrm{WHNF_C} \quad \langle \mathsf{M}, l\rangle \to \langle \mathsf{M}', l\rangle}{\langle \mathsf{M} \cdot \mathsf{N}, l\rangle \to \langle \mathsf{M}' \cdot \mathsf{N}, l\rangle}\ (\mu 1)$$

$$\frac{}{\langle \overline{n}, l\rangle \to \langle \lfloor l - n\rfloor, l\rangle}\ (\mathsf{Par}) \qquad \frac{\mathsf{M} \in \mathrm{WHNF_C} \quad \mathsf{M} \not\equiv \lambda\!\!\lambda.\mathsf{B} \quad \langle \mathsf{M}, l\rangle \to \langle \mathsf{M}', l\rangle}{\langle \mathsf{M} \cdot \mathsf{N}, l\rangle \to \langle \mathsf{M}' \cdot \mathsf{N}, l\rangle}\ (\mu 2)$$

$$\frac{n \geq |\rho|}{\langle n[\rho], l\rangle \to \langle \lfloor n - (|\rho| - l)\rfloor, l\rangle}\ (\mathsf{Fre}) \qquad \frac{\mathsf{M} \in \mathrm{NF_C} \quad \mathsf{M} \not\equiv \lambda\!\!\lambda.\mathsf{B} \quad \langle \mathsf{N}, l\rangle \to \langle \mathsf{N}', l\rangle}{\langle \mathsf{M} \cdot \mathsf{N}, l\rangle \to \langle \mathsf{M} \cdot \mathsf{N}', l\rangle}\ (\nu)$$

$$\frac{}{\langle (\lambda.B)[\rho], l\rangle \to \langle \lambda\!\!\lambda.B[\overline{l+1} : \rho], l\rangle}\ (\mathsf{Lam}) \qquad \frac{\langle \mathsf{B}, l+1\rangle \to \langle \mathsf{B}', l+1\rangle}{\langle \lambda\!\!\lambda.\mathsf{B}, l\rangle \to \langle \lambda\!\!\lambda.\mathsf{B}', l\rangle}\ (\xi)$$

$$\begin{array}{rcl} \mathrm{WHNF_C} & ::= & \lambda\!\!\lambda.\mathsf{C} \mid \lfloor n\rfloor\{\cdot\mathsf{C}\}^* \\ \mathrm{NF_C} & ::= & \lambda\!\!\lambda.\mathrm{NF_C} \mid \lfloor n\rfloor\{\cdot\mathrm{NF_C}\}^* \quad (\simeq \mathrm{NF}) \end{array}$$

Natural Semantics with Explicit Control (λ^*_ρ) $\quad \mathbf{c} ::= \mathbf{n} \mid \mathbf{w}$

$$\frac{n < |\rho| \quad \langle n^{\text{th}}(\rho), l, \mathbf{c}\rangle \Downarrow \mathrm{N}}{\langle n[\rho], l, \mathbf{c}\rangle \Downarrow \mathrm{N}}\ (\mathrm{Var}) \qquad \frac{}{\langle \overline{n}, l, \mathbf{c}\rangle \Downarrow \lfloor l - n \rfloor}\ (\mathrm{Par})$$

$$\frac{n \geq |\rho|}{\langle n[\rho], l, \mathbf{c}\rangle \Downarrow \lfloor n - (|\rho| - l) \rfloor}\ (\mathrm{Fre})$$

$$\frac{}{\langle (\lambda.B)[\rho], l, \mathbf{w}\rangle \Downarrow B[\overline{l+1} : \rho]}\ (\mathrm{Lam1})$$

$$\frac{\langle B[\overline{l+1} : \rho], l+1, \mathbf{n}\rangle \Downarrow \lfloor B' \rfloor}{\langle (\lambda.B)[\rho], l, \mathbf{n}\rangle \Downarrow \lfloor \lambda.B' \rfloor}\ (\mathrm{Lam2})$$

$$\frac{\langle M[\rho], l, \mathbf{w}\rangle \Downarrow \mathrm{M}' \quad \mathrm{M}' \equiv B[\overline{n} : \rho] \quad \langle B[N[\rho] : \rho], l, \mathbf{c}\rangle \Downarrow \mathrm{B}'}{\langle (M\ N)[\rho], l, \mathbf{c}\rangle \Downarrow \mathrm{B}'}\ (\mathrm{Red})$$

$$\frac{\langle M[\rho], l, \mathbf{w}\rangle \Downarrow \mathrm{M}' \quad \mathrm{M}' \equiv \lfloor M' \rfloor \quad \langle N[\rho], l, \mathbf{n}\rangle \Downarrow \lfloor N' \rfloor}{\langle (M\ N)[\rho], l, \mathbf{c}\rangle \Downarrow \lfloor M'\ N' \rfloor}\ (\mathrm{Neu})$$

Optimised Full-Reducing Krivine Machine ($\lambda^*_{\overline{\rho}}$)

$$
\begin{array}{rcl}
\mathsf{C} & ::= & \Lambda[\rho] \mid \overline{n} \mid \lfloor \Lambda \rfloor \\
\rho & ::= & \epsilon \mid \mathsf{C} : \rho
\end{array}
$$

$$
S \quad ::= \quad \epsilon \mid \Lambda[\rho] : S \mid \lambda\!\!\lambda : S \mid \lfloor \Lambda \rfloor : S
$$

$$
\begin{array}{|rcl|}
\hline
T & \to & (\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ ) \\
\hline
(\ (n+1)[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ ) & \to & (\ n[\rho]\ ,\ S\ ,\ l\ ) \\
(\ 0[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ ) & \to & (\ \mathsf{C}\ ,\ S\ ,\ l\ ) \\
(\ n[\epsilon]\ ,\ S\ ,\ l\ ) & \to & (\ \lfloor n + l \rfloor\ ,\ S\ ,\ l\ ) \\
(\ (M\ N)[\rho]\ ,\ S\ ,\ l\ ) & \to & (\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ ) \\
(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ ) & \to & (\ B[\overline{N[\rho']} : \rho]\ ,\ S\ ,\ l\ ) \\
(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ ) & \to & (\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ ) \\
(\ \overline{n}\ ,\ S\ ,\ l\ ) & \to & (\ \lfloor l - n \rfloor\ ,\ S\ ,\ l\ ) \\
(\ \lfloor M \rfloor\ ,\ N[\rho] : S\ ,\ l\ ) & \to & (\ N[\rho]\ ,\ \lfloor M \rfloor : S\ ,\ l\ ) \\
(\ \lfloor B \rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ ) & \to & (\ \lfloor \lambda.B \rfloor\ ,\ S\ ,\ l-1\ ) \\
(\ \lfloor N \rfloor\ ,\ \lfloor M \rfloor : S\ ,\ l\ ) & \to & (\ \lfloor M\ N \rfloor\ ,\ S\ ,\ l\ ) \\
\hline
(\ \lfloor T \rfloor\ ,\ \epsilon\ ,\ 0\ ) & \to & T \\
\hline
\end{array}
$$

Optimised Full-Reducing Krivine Machine ($\lambda_{\overline{\rho}}^{*}$)

$$
\begin{array}{rcl}
C & ::= & \Lambda[\rho] \mid \overline{n} \mid \lfloor \Lambda \rfloor \\
\rho & ::= & \epsilon \mid C : \rho \\
 & & \\
S & ::= & \epsilon \mid \Lambda[\rho] : S \mid \lambda\!\!\lambda : S \mid \lfloor \Lambda \rfloor : S
\end{array}
$$

T	$\to$	$(\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ )$
$(\ (n+1)[C : \rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ n[\rho]\ ,\ S\ ,\ l\ )$
$(\ 0[C : \rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ C\ ,\ S\ ,\ l\ )$
$(\ n[\epsilon]\ ,\ S\ ,\ l\ )$	$\to$	$(\ \lfloor n + l \rfloor\ ,\ S\ ,\ l\ )$
$(\ (M\ N)[\rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ )$	$\to$	$(\ B[N[\rho'] : \rho]\ ,\ S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ )$
$(\ \overline{n}\ ,\ S\ ,\ l\ )$	$\to$	$(\ \lfloor l - n \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor M \rfloor\ ,\ N[\rho] : S\ ,\ l\ )$	$\to$	$(\ N[\rho]\ ,\ \lfloor M \rfloor : S\ ,\ l\ )$
$(\ \lfloor B \rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ )$	$\to$	$(\ \lfloor \lambda.B \rfloor\ ,\ S\ ,\ l-1\ )$
$(\ \lfloor N \rfloor\ ,\ \lfloor M \rfloor : S\ ,\ l\ )$	$\to$	$(\ \lfloor M\ N \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor T \rfloor\ ,\ \epsilon\ ,\ 0\ )$	$\to$	T

Optimised Full-Reducing Krivine Machine ($\lambda^*_{\overline{\rho}}$)

$$
\begin{array}{rcl}
C & ::= & \Lambda[\rho] \mid \overline{n} \mid \lfloor \Lambda \rfloor \\
\rho & ::= & \epsilon \mid C : \rho \\
\\
S & ::= & \epsilon \mid \Lambda[\rho] : S \mid \lambda\!\!\lambda : S \mid \lfloor \Lambda \rfloor : S
\end{array}
$$

T	$\to$	$(\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ )$
$(\ (n+1)[C : \rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ n[\rho]\ ,\ S\ ,\ l\ )$
$(\ 0[C : \rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ C\ ,\ S\ ,\ l\ )$
$(\ n[\epsilon]\ ,\ S\ ,\ l\ )$	$\to$	$(\ \lfloor n + l \rfloor\ ,\ S\ ,\ l\ )$
$(\ (M\ N)[\rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ )$	$\to$	$(\ B[\overline{N[\rho']} : \rho]\ ,\ S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ )$
$(\ \overline{n}\ ,\ S\ ,\ l\ )$	$\to$	$(\ \lfloor l - n \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor M \rfloor\ ,\ N[\rho] : S\ ,\ l\ )$	$\to$	$(\ N[\rho]\ ,\ \lfloor M \rfloor : S\ ,\ l\ )$
$(\ \lfloor B \rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ )$	$\to$	$(\ \lfloor \lambda.B \rfloor\ ,\ S\ ,\ l-1\ )$
$(\ \lfloor N \rfloor\ ,\ \lfloor M \rfloor : S\ ,\ l\ )$	$\to$	$(\ \lfloor M\ N \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor T \rfloor\ ,\ \epsilon\ ,\ 0\ )$	$\to$	T

Optimised Full-Reducing Krivine Machine ($\lambda^{*}_{\overline{\rho}}$)

$$
\begin{array}{rcl}
\mathsf{C} & ::= & \Lambda[\rho] \mid \overline{n} \mid \lfloor \Lambda \rfloor \\
\rho & ::= & \epsilon \mid \mathsf{C} : \rho
\end{array}
$$

$$
S \quad ::= \quad \epsilon \mid \Lambda[\rho] : S \mid \lambda\!\!\lambda : S \mid \lfloor \Lambda \rfloor : S
$$

$$
\begin{array}{|rcl|}
\hline
T & \to & (\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ ) \\
\hline
(\ (n+1)[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ ) & \to & (\ n[\rho]\ ,\ S\ ,\ l\ ) \\
(\ 0[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ ) & \to & (\ \mathsf{C}\ ,\ S\ ,\ l\ ) \\
(\ n[\epsilon]\ ,\ S\ ,\ l\ ) & \to & (\ \lfloor n + l \rfloor\ ,\ S\ ,\ l\ ) \\
(\ (M\ N)[\rho]\ ,\ S\ ,\ l\ ) & \to & (\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ ) \\
(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ ) & \to & (\ B[\overline{N[\rho']} : \rho]\ ,\ S\ ,\ l\ ) \\
(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ ) & \to & (\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ ) \\
(\ \overline{n}\ ,\ S\ ,\ l\ ) & \to & (\ \lfloor l - n \rfloor\ ,\ S\ ,\ l\ ) \\
(\ \lfloor M \rfloor\ ,\ N[\rho] : S\ ,\ l\ ) & \to & (\ N[\rho]\ ,\ \lfloor M \rfloor : S\ ,\ l\ ) \\
(\ \lfloor B \rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ ) & \to & (\ \lfloor \lambda.B \rfloor\ ,\ S\ ,\ l-1\ ) \\
(\ \lfloor N \rfloor\ ,\ \lfloor M \rfloor : S\ ,\ l\ ) & \to & (\ \lfloor M\ N \rfloor\ ,\ S\ ,\ l\ ) \\
\hline
(\ \lfloor T \rfloor\ ,\ \epsilon\ ,\ 0\ ) & \to & T \\
\hline
\end{array}
$$

Optimised Full-Reducing Krivine Machine (λ^*_ρ)

$$
\begin{array}{rcl}
\mathsf{C} & ::= & \Lambda[\rho] \mid \overline{n} \mid \lfloor\Lambda\rfloor \\
\rho & ::= & \epsilon \mid \mathsf{C} : \rho \\
\\
S & ::= & \epsilon \mid \Lambda[\rho] : S \mid \lambda\!\!\lambda : S \mid \lfloor\Lambda\rfloor : S
\end{array}
$$

$$
\begin{array}{|rcl|}
\hline
T & \to & (\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ ) \\
\hline
(\ (n+1)[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ ) & \to & (\ n[\rho]\ ,\ S\ ,\ l\ ) \\
(\ 0[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ ) & \to & (\ \mathsf{C}\ ,\ S\ ,\ l\ ) \\
 & & \\
(\ (M\ N)[\rho]\ ,\ S\ ,\ l\ ) & \to & (\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ ) \\
(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ ) & \to & (\ B[N[\rho'] : \rho]\ ,\ S\ ,\ l\ ) \\
(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ ) & \to & (\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ ) \\
(\ \overline{n}\ ,\ S\ ,\ l\ ) & \to & (\ \lfloor l-n\rfloor\ ,\ S\ ,\ l\ ) \\
(\ \lfloor M\rfloor\ ,\ N[\rho] : S\ ,\ l\ ) & \to & (\ N[\rho]\ ,\ \lfloor M\rfloor : S\ ,\ l\ ) \\
(\ \lfloor B\rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ ) & \to & (\ \lfloor\lambda.B\rfloor\ ,\ S\ ,\ l-1\ ) \\
(\ \lfloor N\rfloor\ ,\ \lfloor M\rfloor : S\ ,\ l\ ) & \to & (\ \lfloor M\ N\rfloor\ ,\ S\ ,\ l\ ) \\
\hline
(\ \lfloor T\rfloor\ ,\ \epsilon\ ,\ 0\ ) & \to & T \\
\hline
\end{array}
$$

Optimised Full-Reducing Krivine Machine (λ^*_ρ)

$$
\begin{array}{rcl}
\mathsf{C} & ::= & \Lambda[\rho] \mid \overline{n} \mid \lfloor \Lambda \rfloor \\
\rho & ::= & \epsilon \mid \mathsf{C} : \rho
\end{array}
$$

$$
S \quad ::= \quad \epsilon \mid \Lambda[\rho] : S \mid \lambda\!\!\lambda : S \mid \lfloor \Lambda \rfloor : S
$$

T	$\rightarrow$	$(\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ )$
$(\ (n+1)[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ n[\rho]\ ,\ S\ ,\ l\ )$
$(\ 0[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \mathsf{C}\ ,\ S\ ,\ l\ )$
$(\ (M\ N)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ )$	$\rightarrow$	$(\ B[\overline{N[\rho']} : \rho]\ ,\ S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ )$
$(\ \overline{n}\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor l-n \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor M \rfloor\ ,\ N[\rho] : S\ ,\ l\ )$	$\rightarrow$	$(\ N[\rho]\ ,\ \lfloor M \rfloor : S\ ,\ l\ )$
$(\ \lfloor B \rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor \lambda.B \rfloor\ ,\ S\ ,\ l-1\ )$
$(\ \lfloor N \rfloor\ ,\ \lfloor M \rfloor : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor M\ N \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor T \rfloor\ ,\ \epsilon\ ,\ 0\ )$	$\rightarrow$	T

Optimised Full-Reducing Krivine Machine (λ^*_ρ)

$$
\begin{array}{rcl}
\mathsf{C} & ::= & \Lambda[\rho] \mid \overline{n} \mid \lfloor \Lambda \rfloor \\
\rho & ::= & \epsilon \mid \mathsf{C} : \rho
\end{array}
$$

$$
S \;::=\; \epsilon \mid \Lambda[\rho] : S \mid \lambda\!\!\lambda : S \mid \lfloor \Lambda \rfloor : S
$$

T	$\rightarrow$	$(\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ )$
$(\ (n+1)[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ n[\rho]\ ,\ S\ ,\ l\ )$
$(\ 0[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \mathsf{C}\ ,\ S\ ,\ l\ )$
$(\ (M\ N)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ )$	$\rightarrow$	$(\ B[N[\rho'] : \rho]\ ,\ S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ )$
$(\ \overline{n}\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor l-n \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor M \rfloor\ ,\ N[\rho] : S\ ,\ l\ )$	$\rightarrow$	$(\ N[\rho]\ ,\ \lfloor M \rfloor : S\ ,\ l\ )$
$(\ \lfloor B \rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor \lambda.B \rfloor\ ,\ S\ ,\ \color{red}{l-1}\ )$
$(\ \lfloor N \rfloor\ ,\ \lfloor M \rfloor : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor M\ N \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor T \rfloor\ ,\ \epsilon\ ,\ 0\ )$	$\rightarrow$	T

Optimised Full-Reducing Krivine Machine (λ^*_{ρ})

$$
\begin{array}{rcl}
\mathsf{C} & ::= & \Lambda[\rho] \mid \overline{n} \mid \lfloor \Lambda \rfloor \\
\rho & ::= & \epsilon \mid \mathsf{C} : \rho \\
\\
S & ::= & \epsilon \mid \Lambda[\rho] : S \mid \lambda\!\!\lambda : S \mid \lfloor \Lambda \rfloor : S
\end{array}
$$

T	$\rightarrow$	$(\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ )$
$(\ (n+1)[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ n[\rho]\ ,\ S\ ,\ l\ )$
$(\ 0[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \mathsf{C}\ ,\ S\ ,\ l\ )$
$(\ (M\ N)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ )$	$\rightarrow$	$(\ B[N[\rho'] : \rho]\ ,\ S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ )$
$(\ \overline{n}\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor l-n \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor M \rfloor\ ,\ N[\rho] : S\ ,\ l\ )$	$\rightarrow$	$(\ N[\rho]\ ,\ \lfloor M \rfloor : S\ ,\ l\ )$
$(\ \lfloor B \rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor \lambda.B \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor N \rfloor\ ,\ \lfloor M \rfloor : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor M\ N \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor T \rfloor\ ,\ \epsilon\ ,\ 0\ )$	$\rightarrow$	T

Optimised Full-Reducing Krivine Machine (λ^*_ρ)

$$
\begin{array}{rcl}
\mathsf{C} & ::= & \Lambda[\rho] \mid \overline{n} \mid \lfloor\Lambda\rfloor \\
\rho & ::= & \epsilon \mid \mathsf{C} : \rho
\end{array}
$$

$$
S \quad ::= \quad \epsilon \mid \Lambda[\rho] : S \mid \lambda\!\!\lambda : S \mid \lfloor\Lambda\rfloor : S
$$

T	$\rightarrow$	$(\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ )$
$(\ (n+1)[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ n[\rho]\ ,\ S\ ,\ l\ )$
$(\ 0[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \mathsf{C}\ ,\ S\ ,\ l\ )$
$(\ (M\ N)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ )$	$\rightarrow$	$(\ B[N[\rho'] : \rho]\ ,\ S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ )$
$(\ \overline{n}\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor l-n\rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor M\rfloor\ ,\ N[\rho] : S\ ,\ l\ )$	$\rightarrow$	$(\ N[\rho]\ ,\ \lfloor M\rfloor : S\ ,\ l\ )$
$(\ \lfloor B\rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor\lambda.B\rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor N\rfloor\ ,\ \lfloor M\rfloor : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor M\ N\rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor T\rfloor\ ,\ \epsilon\ ,\ 0\ )$	$\rightarrow$	T

$$
n(\lambda.B)N
$$

Optimised Full-Reducing Krivine Machine (λ^*_ρ)

$$
\begin{array}{lcl}
\mathsf{C} & ::= & \Lambda[\rho] \mid \overline{n} \mid \lfloor \Lambda, l \rfloor \\
\rho & ::= & \epsilon \mid \mathsf{C} : \rho
\end{array}
$$

$$
\begin{array}{lcl}
S & ::= & \epsilon \mid \Lambda[\rho] : S \mid \lambda\!\!\lambda : S \mid \lfloor \Lambda, l \rfloor : S
\end{array}
$$

$$
\begin{array}{rcl}
\hline
T & \to & (\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ ) \\
\hline
(\ (n+1)[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ ) & \to & (\ n[\rho]\ ,\ S\ ,\ l\ ) \\
(\ 0[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ ) & \to & (\ \mathsf{C}\ ,\ S\ ,\ l\ ) \\
(\ (M\ N)[\rho]\ ,\ S\ ,\ l\ ) & \to & (\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ ) \\
(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ ) & \to & (\ B[\overline{N[\rho']} : \rho]\ ,\ S\ ,\ l\ ) \\
(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ ) & \to & (\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ ) \\
(\ \overline{n}\ ,\ S\ ,\ l\ ) & \to & (\ \lfloor l-n, l \rfloor\ ,\ S\ ,\ l\ ) \\
(\ \lfloor M, l' \rfloor\ ,\ N[\rho] : S\ ,\ l\ ) & \to & (\ N[\rho]\ ,\ \lfloor M, l' \rfloor : S\ ,\ l\ ) \\
(\ \lfloor B, l' \rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ ) & \to & (\ \lfloor \lambda.B, l' \rfloor\ ,\ S\ ,\ l\ ) \\
(\ \lfloor N, l' \rfloor\ ,\ \lfloor M, l'' \rfloor : S\ ,\ l\ ) & \to & (\ \lfloor M\ N, l'' \rfloor\ ,\ S\ ,\ l\ ) \\
\hline
(\ \lfloor T, l' \rfloor\ ,\ \epsilon\ ,\ 0\ ) & \to & T \\
\hline
\end{array}
$$

Optimised Full-Reducing Krivine Machine (λ^*_ρ)

$$
\begin{array}{lcl}
\mathsf{C} & ::= & \Lambda[\rho] \mid \overline{n} \mid \lfloor \Lambda, l \rfloor \\
\rho & ::= & \epsilon \mid \mathsf{C} : \rho
\end{array}
$$

$$
S \quad ::= \quad \epsilon \mid \Lambda[\rho] : S \mid \lambda\!\!\lambda : S \mid \lfloor \Lambda, l \rfloor : S
$$

T	$\rightarrow$	$(\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ )$
$(\ (n+1)[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ n[\rho]\ ,\ S\ ,\ l\ )$
$(\ 0[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \mathsf{C}\ ,\ S\ ,\ l\ )$
$(\ (M\ N)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ )$	$\rightarrow$	$(\ B[\overline{N[\rho']} : \rho]\ ,\ S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ )$
$(\ \overline{n}\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor l-n, l \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor M, l' \rfloor\ ,\ N[\rho] : S\ ,\ l\ )$	$\rightarrow$	$(\ N[\rho]\ ,\ \lfloor M, l' \rfloor : S\ ,\ l\ )$
$(\ \lfloor B, l' \rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor \lambda.B, l' \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor N, l' \rfloor\ ,\ \lfloor M, l'' \rfloor : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor M\ N, l'' \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor T, l' \rfloor\ ,\ \epsilon\ ,\ 0\ )$	$\rightarrow$	T

Optimised Full-Reducing Krivine Machine (λ^*_ρ)

$$
\begin{array}{rcl}
\mathsf{C} & ::= & \Lambda[\rho] \mid \overline{n} \mid \lfloor \Lambda, l \rfloor \\
\rho & ::= & \epsilon \mid \mathsf{C} : \rho \\
\\
S & ::= & \epsilon \mid \Lambda[\rho] : S \mid \lambda\!\!\lambda : S \mid \lfloor \Lambda, l \rfloor : S
\end{array}
$$

T	$\to$	$(\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ )$
$(\ (n+1)[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ n[\rho]\ ,\ S\ ,\ l\ )$
$(\ 0[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ \mathsf{C}\ ,\ S\ ,\ l\ )$
$(\ (M\ N)[\rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ )$	$\to$	$(\ B[\overline{N[\rho']} : \rho]\ ,\ S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ )$
$(\ \overline{n}\ ,\ S\ ,\ l\ )$	$\to$	$(\ \lfloor l-n, l \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor M, l' \rfloor\ ,\ N[\rho] : S\ ,\ l\ )$	$\to$	$(\ N[\rho]\ ,\ \lfloor M, l' \rfloor : S\ ,\ l\ )$
$(\ \lfloor B, l' \rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ )$	$\to$	$(\ \lfloor \lambda.B, l' \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor N, l' \rfloor\ ,\ \lfloor M, l'' \rfloor : S\ ,\ l\ )$	$\to$	$(\ \lfloor M\ N, l'' \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor T, l' \rfloor\ ,\ \epsilon\ ,\ 0\ )$	$\to$	T

Optimised Full-Reducing Krivine Machine (λ^*_ρ)

$$
\begin{array}{rcl}
\mathsf{C} & ::= & \Lambda[\rho] \mid \overline{n} \mid \lfloor \Lambda, l \rfloor \\
\rho & ::= & \epsilon \mid \mathsf{C} : \rho
\end{array}
$$

$$
\begin{array}{rcl}
S & ::= & \epsilon \mid \Lambda[\rho] : S \mid \lambda\!\!\lambda : S \mid \lfloor \Lambda, l \rfloor : S
\end{array}
$$

T	$\to$	$(\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ )$
$(\ (n+1)[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ n[\rho]\ ,\ S\ ,\ l\ )$
$(\ 0[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ \mathsf{C}\ ,\ S\ ,\ l\ )$
$(\ (M\ N)[\rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ )$	$\to$	$(\ B[\overline{N[\rho']} : \rho]\ ,\ S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ )$
$(\ \overline{n}\ ,\ S\ ,\ l\ )$	$\to$	$(\ \lfloor l-n, l \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor M, l' \rfloor\ ,\ N[\rho] : S\ ,\ l\ )$	$\to$	$(\ N[\rho]\ ,\ \lfloor M, l' \rfloor : S\ ,\ l'\ )$
$(\ \lfloor B, l' \rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ )$	$\to$	$(\ \lfloor \lambda.B, l' \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor N, l' \rfloor\ ,\ \lfloor M, l'' \rfloor : S\ ,\ l\ )$	$\to$	$(\ \lfloor M\ N, l'' \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor T, l' \rfloor\ ,\ \epsilon\ ,\ 0\ )$	$\to$	T

Optimised Full-Reducing Krivine Machine (λ^*_ρ)

$$
\begin{array}{rcl}
\mathsf{C} & ::= & \Lambda[\rho] \mid \overline{n} \mid \lfloor \Lambda, l \rfloor \\
\rho & ::= & \epsilon \mid \mathsf{C} : \rho \\
 & & \\
S & ::= & \epsilon \mid \Lambda[\rho] : S \mid \lambda\!\!\lambda : S \mid \lfloor \Lambda, l \rfloor : S
\end{array}
$$

$$
\begin{array}{|rcl|}
\hline
T & \to & (\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ ) \\
\hline
(\ (n+1)[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ ) & \to & (\ n[\rho]\ ,\ S\ ,\ l\ ) \\
(\ 0[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ ) & \to & (\ \mathsf{C}\ ,\ S\ ,\ l\ ) \\
(\ (M\ N)[\rho]\ ,\ S\ ,\ l\ ) & \to & (\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ ) \\
(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ ) & \to & (\ B[N[\rho'] : \rho]\ ,\ S\ ,\ l\ ) \\
(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ ) & \to & (\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ ) \\
(\ \overline{n}\ ,\ S\ ,\ l\ ) & \to & (\ \lfloor l-n, l \rfloor\ ,\ S\ ,\ l\ ) \\
(\ \lfloor M, l' \rfloor\ ,\ N[\rho] : S\ ,\ l\ ) & \to & (\ N[\rho]\ ,\ \lfloor M, l' \rfloor : S\ ,\ l'\ ) \\
(\ \lfloor B, l' \rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ ) & \to & (\ \lfloor \lambda.B, l' \rfloor\ ,\ S\ ,\ l\ ) \\
(\ \lfloor N, l' \rfloor\ ,\ \lfloor M, l'' \rfloor : S\ ,\ l\ ) & \to & (\ \lfloor M\ N, l'' \rfloor\ ,\ S\ ,\ l\ ) \\
\hline
(\ \lfloor T, l' \rfloor\ ,\ \epsilon\ ,\ 0\ ) & \to & T \\
\hline
\end{array}
$$

Original Full-Reducing Krivine Machine $\lambda_{\overline{\rho}}$

$$
\begin{array}{rcl}
\mathsf{C} & ::= & \Lambda[\rho] \mid \overline{n} \mid \lfloor \Lambda, l \rfloor \\
\rho & ::= & \epsilon \mid \mathsf{C} : \rho \\
 & & \\
S & ::= & \epsilon \mid \Lambda[\rho] : S \mid \lambda\!\!\lambda : S \mid \lfloor \Lambda, l \rfloor : S
\end{array}
$$

$$
\begin{array}{|rcl|}
\hline
T & \to & (\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ ) \\
\hline
(\ (n+1)[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ ) & \to & (\ n[\rho]\ ,\ S\ ,\ l\ ) \\
(\ 0[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ ) & \to & (\ \mathsf{C}\ ,\ S\ ,\ l\ ) \\
(\ (M\ N)[\rho]\ ,\ S\ ,\ l\ ) & \to & (\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ ) \\
(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ ) & \to & (\ B[\overline{N[\rho']} : \rho]\ ,\ S\ ,\ l\ ) \\
(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ ) & \to & (\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ ) \\
(\ \overline{n}\ ,\ S\ ,\ l\ ) & \to & (\ \lfloor l-n, l \rfloor\ ,\ S\ ,\ l\ ) \\
(\ \lfloor M, l' \rfloor\ ,\ N[\rho] : S\ ,\ l\ ) & \to & (\ N[\rho]\ ,\ \lfloor M, l' \rfloor : S\ ,\ l'\ ) \\
(\ \lfloor B, l' \rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ ) & \to & (\ \lfloor \lambda.B, l' \rfloor\ ,\ S\ ,\ l\ ) \\
(\ \lfloor N, l' \rfloor\ ,\ \lfloor M, l'' \rfloor : S\ ,\ l\ ) & \to & (\ \lfloor M\ N, l'' \rfloor\ ,\ S\ ,\ l\ ) \\
\hline
(\ \lfloor T, l' \rfloor\ ,\ \epsilon\ ,\ l\ ) & \to & T \\
\hline
\end{array}
$$

Summary

- **Structural operational semantics (small-step):** exposes ephemeral constructs, deBruijn levels representing parameters ($\overline{n}$).

Summary

- **Structural operational semantics (small-step):** exposes ephemeral constructs, deBruijn levels representing parameters ($\overline{n}$).
- **Natural semantics (big-step)**: control character distinguishes between weak ($\mathbf{w}$) and full ($\mathbf{n}$) evaluation.

Summary

- **Structural operational semantics (small-step):** exposes ephemeral constructs, deBruijn levels representing parameters ($\overline{n}$).
- **Natural semantics (big-step)**: control character distinguishes between weak (**w**) and full (**n**) evaluation.
- **Abstract machine (first-order state-transition function)**: the distinction is done looking at the control stack.

Calculi of Closures Hierarchy

$$C ::= \Lambda[\rho]$$

$$\begin{tikzcd}[column sep=small]
 & \lambda_{\rho}\,\text{[Curien, 1991]} \arrow[dl, "\begin{array}{l}+\overline{n}\\ +\lfloor\Lambda\rfloor\end{array}"'] \arrow[dr, "+C\cdot C"] & \\
\lambda^{*}_{\overline{\rho}}\ \text{based on [Crégut, 2007]} \arrow[dr, "\begin{array}{l}\lfloor\Lambda\rfloor \simeq \lfloor n\rfloor\\ \quad\mid \lambda\!\!\lambda.C\\ \quad\mid C\cdot C\end{array}"'] & & \lambda_{\widehat{\rho}}\ \text{[Biernacka and Danvy, 2007]} \arrow[dl, "\begin{array}{l}+\overline{n}\\ +\lfloor n\rfloor\\ +\lambda\!\!\lambda.C\end{array}"] \\
 & \lambda_{\widetilde{\rho}} &
\end{tikzcd}$$

Calculi of Closures Hierarchy

$C ::= \Lambda[\rho]$

$$
\begin{tikzcd}
 & \lambda_{\rho}\ \text{[Curien, 1991]} \arrow[dl, "{+\overline{n}}" ', "{+ \lfloor \Lambda \rfloor}" '] \arrow[dr, "{+C \cdot C}"] & \\
\lambda^{*}_{\overline{\rho}}\ \text{based on [Crégut, 2007]} \arrow[dr, "{\lfloor \Lambda \rfloor \simeq \lfloor n \rfloor \mid \lambda\!\!\lambda.C \mid C \cdot C}" '] & & \lambda_{\widehat{\rho}}\ \text{[Biernacka and Danvy, 2007]} \arrow[dl, "{+\overline{n} \; + \lfloor n \rfloor \; + \lambda\!\!\lambda.C}"] \\
 & \lambda_{\widetilde{\rho}} &
\end{tikzcd}
$$

Thanks!

Closure-Based Substitution

$$
\begin{array}{rcl}
\sigma_{\mathsf{c}}(\lfloor n \rfloor, l) & = & n \\
\sigma_{\mathsf{c}}(n[\rho], l) & = & \begin{cases} \sigma_{\mathsf{c}}(n^{\text{th}}(\rho), l) & \text{if } n < |\rho| \\ n - (|\rho| - l) & \text{if } n \geq |\rho| \end{cases} \\
\sigma_{\mathsf{c}}((\lambda.B)[\rho], l) & = & \sigma_{\mathsf{c}}(\lambda\!\!\lambda.B[\overline{l+1} : \rho], l) \\
\sigma_{\mathsf{c}}(\lambda\!\!\lambda.\mathrm{B}, l) & = & \lambda.\sigma_{\mathsf{c}}(\mathrm{B}, l+1) \\
\sigma_{\mathsf{c}}(\overline{n}, l) & = & l - n \\
\sigma_{\mathsf{c}}((M\ N)[\rho], l) & = & \sigma_{\mathsf{c}}(M[\rho] \cdot N[\rho], l) \\
\sigma_{\mathsf{c}}(\mathrm{M} \cdot \mathrm{N}, l) & = & \sigma_{\mathsf{c}}(\mathrm{M}, l)\, \sigma_{\mathsf{c}}(\mathrm{N}, l)
\end{array}
$$

Step-by-Step Simulation between $\lambda_{\widetilde{\rho}}$ and λ

$$\mathsf{M} \equiv M[\epsilon]$$

$$\begin{tikzcd}
\mathsf{M} \arrow[r, "*", "\rho"'] & \mathsf{M}' \arrow[r, "\beta"'] & \mathsf{M}_1 \arrow[r, "*", "\rho"'] \arrow[d, "\sigma"'] & \mathsf{M}_1' \arrow[r, "\beta"'] & \mathsf{M}_2 \arrow[d, "\sigma"'] & \cdots\cdots \\
M \arrow[u, hook] \arrow[rr] & & M_1 \arrow[rr] & & M_2 & \cdots\cdots
\end{tikzcd}$$

$\lambda_{\widetilde{\rho}}$	$\to_\rho$	all the rules but β
	$\to_\beta$	β and all the compatibility rules
	$\to_\sigma$	closure-based substitution
λ	$\to$	substitution-based normal order

Height of a Closure

$$
\begin{array}{rcl}
h(n[\rho]) & = & \left\{ \begin{array}{ll} h(n^{\mathrm{th}}(\rho)) & \text{if } n < |\rho| \\ 0 & \text{if } n \geq |\rho| \end{array} \right. \\
h((\lambda.B)[\rho]) & = & 1 + h(B[\overline{n} : \rho]) \\
h((M\ N)[\rho]) & = & \max\{h(M[\rho]), h(N[\rho])\} \\
h(\overline{n}) & = & 0 \\
h(\lfloor n \rfloor) & = & 0 \\
h(\lambda\!\!\lambda.\mathrm{B}) & = & 1 + h(\mathrm{B}) \\
h(\mathrm{M} \cdot \mathrm{N}) & = & \max\{h(\mathrm{M}), h(\mathrm{N})\}
\end{array}
$$

Example with Optimised Machine

$$\lambda.0((\lambda.0)0)$$

T	$\to$	$(\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ )$
$(\ (n+1)[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ n[\rho]\ ,\ S\ ,\ l\ )$
$(\ n[\epsilon]\ ,\ S\ ,\ l\ )$	$\to$	$(\ \lfloor n+l \rfloor\ ,\ S\ ,\ l\ )$
$(\ 0[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ \mathsf{C}\ ,\ S\ ,\ l\ )$
$(\ (M\ N)[\rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ )$	$\to$	$(\ B[\overline{N[\rho']} : \rho]\ ,\ S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ )$
$(\ \overline{n}\ ,\ S\ ,\ l\ )$	$\to$	$(\ \lfloor l-n \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor M \rfloor\ ,\ N[\rho] : S\ ,\ l\ )$	$\to$	$(\ N[\rho]\ ,\ \lfloor M \rfloor : S\ ,\ l\ )$
$(\ \lfloor B \rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ )$	$\to$	$(\ \lfloor \lambda.B \rfloor\ ,\ S\ ,\ l-1\ )$
$(\ \lfloor N \rfloor\ ,\ \lfloor M \rfloor : S\ ,\ l\ )$	$\to$	$(\ \lfloor M\ N \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor T \rfloor\ ,\ \epsilon\ ,\ 0\ )$	$\to$	T

Example with Optimised Machine

$$\lambda.0((\lambda.0)0)$$

T	$\rightarrow$	$(\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ )$
$(\ (n+1)[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ n[\rho]\ ,\ S\ ,\ l\ )$
$(\ n[\epsilon]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor n+l \rfloor\ ,\ S\ ,\ l\ )$
$(\ 0[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \mathsf{C}\ ,\ S\ ,\ l\ )$
$(\ (M\ N)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ )$	$\rightarrow$	$(\ B[N[\rho'] : \rho]\ ,\ S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ )$
$(\ \overline{n}\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor l-n \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor M \rfloor\ ,\ N[\rho] : S\ ,\ l\ )$	$\rightarrow$	$(\ N[\rho]\ ,\ \lfloor M \rfloor : S\ ,\ l\ )$
$(\ \lfloor B \rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor \lambda.B \rfloor\ ,\ S\ ,\ l-1\ )$
$(\ \lfloor N \rfloor\ ,\ \lfloor M \rfloor : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor M\ N \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor T \rfloor\ ,\ \epsilon\ ,\ 0\ )$	$\rightarrow$	T

Example with Optimised Machine

$$(\ \lambda.0((\lambda.0)0)[\epsilon] \ , \ \epsilon \ , \ 0 \ )$$

T	$\to$	$(\ T[\epsilon] \ , \ \epsilon \ , \ 0 \ )$
$(\ (n+1)[\mathsf{C} : \rho] \ , \ S \ , \ l \ )$	$\to$	$(\ n[\rho] \ , \ S \ , \ l \ )$
$(\ n[\epsilon] \ , \ S \ , \ l \ )$	$\to$	$(\ \lfloor n+l \rfloor \ , \ S \ , \ l \ )$
$(\ 0[\mathsf{C} : \rho] \ , \ S \ , \ l \ )$	$\to$	$(\ \mathsf{C} \ , \ S \ , \ l \ )$
$(\ (M\,N)[\rho] \ , \ S \ , \ l \ )$	$\to$	$(\ M[\rho] \ , \ N[\rho] : S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ N[\rho'] : S \ , \ l \ )$	$\to$	$(\ B[N[\rho'] : \rho] \ , \ S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ S \ , \ l \ )$	$\to$	$(\ B[\overline{l+1} : \rho] \ , \ \lambda\!\!\lambda : S \ , \ l+1 \ )$
$(\ \overline{n} \ , \ S \ , \ l \ )$	$\to$	$(\ \lfloor l-n \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor M \rfloor \ , \ N[\rho] : S \ , \ l \ )$	$\to$	$(\ N[\rho] \ , \ \lfloor M \rfloor : S \ , \ l \ )$
$(\ \lfloor B \rfloor \ , \ \lambda\!\!\lambda : S \ , \ l \ )$	$\to$	$(\ \lfloor \lambda.B \rfloor \ , \ S \ , \ l-1 \ )$
$(\ \lfloor N \rfloor \ , \ \lfloor M \rfloor : S \ , \ l \ )$	$\to$	$(\ \lfloor M\,N \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor T \rfloor \ , \ \epsilon \ , \ 0 \ )$	$\to$	T

Example with Optimised Machine

$$(\ \lambda.0((\lambda.0)0)[\epsilon] \ , \ \epsilon \ , \ 0 \ )$$

T	$\rightarrow$	$(\ T[\epsilon] \ , \ \epsilon \ , \ 0 \ )$
$(\ (n+1)[\mathsf{C} : \rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ n[\rho] \ , \ S \ , \ l \ )$
$(\ n[\epsilon] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ \lfloor n+l \rfloor \ , \ S \ , \ l \ )$
$(\ 0[\mathsf{C} : \rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ \mathsf{C} \ , \ S \ , \ l \ )$
$(\ (M\ N)[\rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ M[\rho] \ , \ N[\rho] : S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ N[\rho'] : S \ , \ l \ )$	$\rightarrow$	$(\ B[N[\rho'] : \rho] \ , \ S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ B[\overline{l+1} : \rho] \ , \ \lambda\!\!\lambda : S \ , \ l+1 \ )$
$(\ \overline{n} \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ \lfloor l-n \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor M \rfloor \ , \ N[\rho] : S \ , \ l \ )$	$\rightarrow$	$(\ N[\rho] \ , \ \lfloor M \rfloor : S \ , \ l \ )$
$(\ \lfloor B \rfloor \ , \ \lambda\!\!\lambda : S \ , \ l \ )$	$\rightarrow$	$(\ \lfloor \lambda.B \rfloor \ , \ S \ , \ l-1 \ )$
$(\ \lfloor N \rfloor \ , \ \lfloor M \rfloor : S \ , \ l \ )$	$\rightarrow$	$(\ \lfloor M\ N \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor T \rfloor \ , \ \epsilon \ , \ 0 \ )$	$\rightarrow$	T

Example with Optimised Machine

$$(\; 0((\lambda.0)0)[\overline{1}] \; , \; \lambda\!\!\lambda \; , \; 1 \;)$$

T	$\rightarrow$	$(\; T[\epsilon] \; , \; \epsilon \; , \; 0 \;)$
$(\; (n+1)[\mathrm{C} : \rho] \; , \; S \; , \; l \;)$	$\rightarrow$	$(\; n[\rho] \; , \; S \; , \; l \;)$
$(\; n[\epsilon] \; , \; S \; , \; l \;)$	$\rightarrow$	$(\; \lfloor n+l \rfloor \; , \; S \; , \; l \;)$
$(\; 0[\mathrm{C} : \rho] \; , \; S \; , \; l \;)$	$\rightarrow$	$(\; \mathrm{C} \; , \; S \; , \; l \;)$
$(\; (M\,N)[\rho] \; , \; S \; , \; l \;)$	$\rightarrow$	$(\; M[\rho] \; , \; N[\rho] : S \; , \; l \;)$
$(\; (\lambda.B)[\rho] \; , \; N[\rho'] : S \; , \; l \;)$	$\rightarrow$	$(\; B[\overline{N[\rho']} : \rho] \; , \; S \; , \; l \;)$
$(\; (\lambda.B)[\rho] \; , \; S \; , \; l \;)$	$\rightarrow$	$(\; B[\overline{l+1} : \rho] \; , \; \lambda\!\!\lambda : S \; , \; l+1 \;)$
$(\; \overline{n} \; , \; S \; , \; l \;)$	$\rightarrow$	$(\; \lfloor l-n \rfloor \; , \; S \; , \; l \;)$
$(\; \lfloor M \rfloor \; , \; N[\rho] : S \; , \; l \;)$	$\rightarrow$	$(\; N[\rho] \; , \; \lfloor M \rfloor : S \; , \; l \;)$
$(\; \lfloor B \rfloor \; , \; \lambda\!\!\lambda : S \; , \; l \;)$	$\rightarrow$	$(\; \lfloor \lambda.B \rfloor \; , \; S \; , \; l-1 \;)$
$(\; \lfloor N \rfloor \; , \; \lfloor M \rfloor : S \; , \; l \;)$	$\rightarrow$	$(\; \lfloor M\,N \rfloor \; , \; S \; , \; l \;)$
$(\; \lfloor T \rfloor \; , \; \epsilon \; , \; 0 \;)$	$\rightarrow$	T

Example with Optimised Machine

$$(\ 0((\lambda.0)0)[\overline{1}]\ ,\ \lambda\!\!\lambda\ ,\ 1\ )$$

T	$\rightarrow$	$(\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ )$
$(\ (n+1)[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ n[\rho]\ ,\ S\ ,\ l\ )$
$(\ n[\epsilon]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor n+l \rfloor\ ,\ S\ ,\ l\ )$
$(\ 0[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \mathsf{C}\ ,\ S\ ,\ l\ )$
$(\ (M\ N)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ )$	$\rightarrow$	$(\ B[N[\rho'] : \rho]\ ,\ S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ )$
$(\ \overline{n}\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor l-n \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor M \rfloor\ ,\ N[\rho] : S\ ,\ l\ )$	$\rightarrow$	$(\ N[\rho]\ ,\ \lfloor M \rfloor : S\ ,\ l\ )$
$(\ \lfloor B \rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor \lambda.B \rfloor\ ,\ S\ ,\ l-1\ )$
$(\ \lfloor N \rfloor\ ,\ \lfloor M \rfloor : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor M\ N \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor T \rfloor\ ,\ \epsilon\ ,\ 0\ )$	$\rightarrow$	T

Example with Optimised Machine

$$(\ 0[\overline{1}]\ ,\ ((\lambda.0)0)[\overline{1}] : \lambda\!\!\lambda\ ,\ 1\ )$$

T	$\rightarrow$	$(\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ )$
$(\ (n+1)[\mathrm{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ n[\rho]\ ,\ S\ ,\ l\ )$
$(\ n[\epsilon]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor n+l \rfloor\ ,\ S\ ,\ l\ )$
$(\ 0[\mathrm{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \mathrm{C}\ ,\ S\ ,\ l\ )$
$(\ (M\,N)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ )$	$\rightarrow$	$(\ B[\overline{N[\rho']} : \rho]\ ,\ S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ )$
$(\ \overline{n}\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor l-n \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor M \rfloor\ ,\ N[\rho] : S\ ,\ l\ )$	$\rightarrow$	$(\ N[\rho]\ ,\ \lfloor M \rfloor : S\ ,\ l\ )$
$(\ \lfloor B \rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor \lambda.B \rfloor\ ,\ S\ ,\ l-1\ )$
$(\ \lfloor N \rfloor\ ,\ \lfloor M \rfloor : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor M\,N \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor T \rfloor\ ,\ \epsilon\ ,\ 0\ )$	$\rightarrow$	T

Example with Optimised Machine

$$(\ 0[\overline{1}]\ ,\ ((\lambda.0)0)[\overline{1}] : \lambda\!\!\lambda\ ,\ 1\ )$$

T	$\rightarrow$	$(\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ )$
$(\ (n+1)[\mathrm{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ n[\rho]\ ,\ S\ ,\ l\ )$
$(\ n[\epsilon]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor n+l \rfloor\ ,\ S\ ,\ l\ )$
$(\ 0[\mathrm{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \mathrm{C}\ ,\ S\ ,\ l\ )$
$(\ (M\ N)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ )$	$\rightarrow$	$(\ B[\overline{N[\rho']} : \rho]\ ,\ S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ )$
$(\ \overline{n}\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor l-n \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor M \rfloor\ ,\ N[\rho] : S\ ,\ l\ )$	$\rightarrow$	$(\ N[\rho]\ ,\ \lfloor M \rfloor : S\ ,\ l\ )$
$(\ \lfloor B \rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor \lambda.B \rfloor\ ,\ S\ ,\ l-1\ )$
$(\ \lfloor N \rfloor\ ,\ \lfloor M \rfloor : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor M\ N \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor T \rfloor\ ,\ \epsilon\ ,\ 0\ )$	$\rightarrow$	T

Example with Optimised Machine

$$(\ \overline{1}\ ,\ ((\lambda.0)0)[\overline{1}] : \lambda\!\!\lambda\ ,\ 1\ )$$

T	$\rightarrow$	$(\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ )$
$(\ (n+1)[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ n[\rho]\ ,\ S\ ,\ l\ )$
$(\ n[\epsilon]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor n+l \rfloor\ ,\ S\ ,\ l\ )$
$(\ 0[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \mathsf{C}\ ,\ S\ ,\ l\ )$
$(\ (M\ N)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ )$	$\rightarrow$	$(\ B[\overline{N[\rho']} : \rho]\ ,\ S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ )$
$(\ \overline{n}\ ,\ S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor l-n \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor M \rfloor\ ,\ N[\rho] : S\ ,\ l\ )$	$\rightarrow$	$(\ N[\rho]\ ,\ \lfloor M \rfloor : S\ ,\ l\ )$
$(\ \lfloor B \rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor \lambda.B \rfloor\ ,\ S\ ,\ l-1\ )$
$(\ \lfloor N \rfloor\ ,\ \lfloor M \rfloor : S\ ,\ l\ )$	$\rightarrow$	$(\ \lfloor M\ N \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor T \rfloor\ ,\ \epsilon\ ,\ 0\ )$	$\rightarrow$	T

Example with Optimised Machine

$$(\, \overline{1} \, , \; ((\lambda.0)0)[\overline{1}] : \lambda\!\!\lambda \, , \; 1 \,)$$

T	$\rightarrow$	$(\, T[\epsilon] \, , \, \epsilon \, , \, 0 \,)$
$(\, (n+1)[\mathsf{C} : \rho] \, , \, S \, , \, l \,)$	$\rightarrow$	$(\, n[\rho] \, , \, S \, , \, l \,)$
$(\, n[\epsilon] \, , \, S \, , \, l \,)$	$\rightarrow$	$(\, \lfloor n + l \rfloor \, , \, S \, , \, l \,)$
$(\, 0[\mathsf{C} : \rho] \, , \, S \, , \, l \,)$	$\rightarrow$	$(\, \mathsf{C} \, , \, S \, , \, l \,)$
$(\, (M\,N)[\rho] \, , \, S \, , \, l \,)$	$\rightarrow$	$(\, M[\rho] \, , \, N[\rho] : S \, , \, l \,)$
$(\, (\lambda.B)[\rho] \, , \, N[\rho'] : S \, , \, l \,)$	$\rightarrow$	$(\, B[\overline{N[\rho']} : \rho] \, , \, S \, , \, l \,)$
$(\, (\lambda.B)[\rho] \, , \, S \, , \, l \,)$	$\rightarrow$	$(\, B[\overline{l+1} : \rho] \, , \, \lambda\!\!\lambda : S \, , \, l+1 \,)$
$(\, \overline{n} \, , \, S \, , \, l \,)$	$\rightarrow$	$(\, \lfloor l - n \rfloor \, , \, S \, , \, l \,)$
$(\, \lfloor M \rfloor \, , \, N[\rho] : S \, , \, l \,)$	$\rightarrow$	$(\, N[\rho] \, , \, \lfloor M \rfloor : S \, , \, l \,)$
$(\, \lfloor B \rfloor \, , \, \lambda\!\!\lambda : S \, , \, l \,)$	$\rightarrow$	$(\, \lfloor \lambda.B \rfloor \, , \, S \, , \, l-1 \,)$
$(\, \lfloor N \rfloor \, , \, \lfloor M \rfloor : S \, , \, l \,)$	$\rightarrow$	$(\, \lfloor M\,N \rfloor \, , \, S \, , \, l \,)$
$(\, \lfloor T \rfloor \, , \, \epsilon \, , \, 0 \,)$	$\rightarrow$	T

Example with Optimised Machine

$$(\lfloor 0 \rfloor \,,\, ((\lambda.0)0)[\overline{1}] : \lambda\!\!\lambda \,,\, 1)$$

T	$\to$	$(T[\epsilon] , \epsilon , 0)$
$((n+1)[\mathsf{C} : \rho] , S , l)$	$\to$	$(n[\rho] , S , l)$
$(n[\epsilon] , S , l)$	$\to$	$(\lfloor n + l \rfloor , S , l)$
$(0[\mathsf{C} : \rho] , S , l)$	$\to$	(C , S , l)
$((M\,N)[\rho] , S , l)$	$\to$	$(M[\rho] , N[\rho] : S , l)$
$((\lambda.B)[\rho] , N[\rho'] : S , l)$	$\to$	$(B[\overline{N[\rho']} : \rho] , S , l)$
$((\lambda.B)[\rho] , S , l)$	$\to$	$(B[\overline{l+1} : \rho] , \lambda\!\!\lambda : S , l + 1)$
$(\overline{n} , S , l)$	$\to$	$(\lfloor l - n \rfloor , S , l)$
$(\lfloor M \rfloor , N[\rho] : S , l)$	$\to$	$(N[\rho] , \lfloor M \rfloor : S , l)$
$(\lfloor B \rfloor , \lambda\!\!\lambda : S , l)$	$\to$	$(\lfloor \lambda.B \rfloor , S , l - 1)$
$(\lfloor N \rfloor , \lfloor M \rfloor : S , l)$	$\to$	$(\lfloor M\,N \rfloor , S , l)$
$(\lfloor T \rfloor , \epsilon , 0)$	$\to$	T

Example with Optimised Machine

$$(\lfloor 0 \rfloor \ , \ ((\lambda.0)0)[\overline{1}] : \lambda\!\!\lambda \ , \ 1 \ )$$

T	$\rightarrow$	$(\ T[\epsilon] \ , \ \epsilon \ , \ 0 \ )$
$(\ (n+1)[\mathsf{C} : \rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ n[\rho] \ , \ S \ , \ l \ )$
$(\ n[\epsilon] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ \lfloor n + l \rfloor \ , \ S \ , \ l \ )$
$(\ 0[\mathsf{C} : \rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ \mathsf{C} \ , \ S \ , \ l \ )$
$(\ (M \ N)[\rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ M[\rho] \ , \ N[\rho] : S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ N[\rho'] : S \ , \ l \ )$	$\rightarrow$	$(\ B[N[\rho'] : \rho] \ , \ S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ B[\overline{l+1} : \rho] \ , \ \lambda\!\!\lambda : S \ , \ l+1 \ )$
$(\ \overline{n} \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ \lfloor l - n \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor M \rfloor \ , \ N[\rho] : S \ , \ l \ )$	$\rightarrow$	$(\ N[\rho] \ , \ \lfloor M \rfloor : S \ , \ l \ )$
$(\ \lfloor B \rfloor \ , \ \lambda\!\!\lambda : S \ , \ l \ )$	$\rightarrow$	$(\ \lfloor \lambda.B \rfloor \ , \ S \ , \ l - 1 \ )$
$(\ \lfloor N \rfloor \ , \ \lfloor M \rfloor : S \ , \ l \ )$	$\rightarrow$	$(\ \lfloor M \ N \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor T \rfloor \ , \ \epsilon \ , \ 0 \ )$	$\rightarrow$	T

Example with Optimised Machine

$$(\; ((\lambda.0)0)[\overline{1}] \; , \; \lfloor 0 \rfloor : \lambda\!\!\lambda \; , \; 1 \;)$$

T	$\to$	$(\; T[\epsilon] \; , \; \epsilon \; , \; 0 \;)$
$(\; (n+1)[\mathsf{C} : \rho] \; , \; S \; , \; l \;)$	$\to$	$(\; n[\rho] \; , \; S \; , \; l \;)$
$(\; n[\epsilon] \; , \; S \; , \; l \;)$	$\to$	$(\; \lfloor n + l \rfloor \; , \; S \; , \; l \;)$
$(\; 0[\mathsf{C} : \rho] \; , \; S \; , \; l \;)$	$\to$	$(\; \mathsf{C} \; , \; S \; , \; l \;)$
$(\; (M\,N)[\rho] \; , \; S \; , \; l \;)$	$\to$	$(\; M[\rho] \; , \; N[\rho] : S \; , \; l \;)$
$(\; (\lambda.B)[\rho] \; , \; N[\rho'] : S \; , \; l \;)$	$\to$	$(\; B[N[\rho'] : \rho] \; , \; S \; , \; l \;)$
$(\; (\lambda.B)[\rho] \; , \; S \; , \; l \;)$	$\to$	$(\; B[\overline{l+1} : \rho] \; , \; \lambda\!\!\lambda : S \; , \; l+1 \;)$
$(\; \overline{n} \; , \; S \; , \; l \;)$	$\to$	$(\; \lfloor l - n \rfloor \; , \; S \; , \; l \;)$
$(\; \lfloor M \rfloor \; , \; N[\rho] : S \; , \; l \;)$	$\to$	$(\; N[\rho] \; , \; \lfloor M \rfloor : S \; , \; l \;)$
$(\; \lfloor B \rfloor \; , \; \lambda\!\!\lambda : S \; , \; l \;)$	$\to$	$(\; \lfloor \lambda.B \rfloor \; , \; S \; , \; l - 1 \;)$
$(\; \lfloor N \rfloor \; , \; \lfloor M \rfloor : S \; , \; l \;)$	$\to$	$(\; \lfloor M\,N \rfloor \; , \; S \; , \; l \;)$
$(\; \lfloor T \rfloor \; , \; \epsilon \; , \; 0 \;)$	$\to$	T

Example with Optimised Machine

$$(\; ((\lambda.0)0)[\overline{1}] \; , \; \lfloor 0 \rfloor : \lambda\!\!\lambda \; , \; 1 \;)$$

T	$\rightarrow$	$(\; T[\epsilon] \; , \; \epsilon \; , \; 0 \;)$
$(\; (n+1)[\mathsf{C} : \rho] \; , \; S \; , \; l \;)$	$\rightarrow$	$(\; n[\rho] \; , \; S \; , \; l \;)$
$(\; n[\epsilon] \; , \; S \; , \; l \;)$	$\rightarrow$	$(\; \lfloor n+l \rfloor \; , \; S \; , \; l \;)$
$(\; 0[\mathsf{C} : \rho] \; , \; S \; , \; l \;)$	$\rightarrow$	$(\; \mathsf{C} \; , \; S \; , \; l \;)$
$(\; (M\,N)[\rho] \; , \; S \; , \; l \;)$	$\rightarrow$	$(\; M[\rho] \; , \; N[\rho] : S \; , \; l \;)$
$(\; (\lambda.B)[\rho] \; , \; N[\rho'] : S \; , \; l \;)$	$\rightarrow$	$(\; B[N[\rho'] : \rho] \; , \; S \; , \; l \;)$
$(\; (\lambda.B)[\rho] \; , \; S \; , \; l \;)$	$\rightarrow$	$(\; B[\overline{l+1} : \rho] \; , \; \lambda\!\!\lambda : S \; , \; l+1 \;)$
$(\; \overline{n} \; , \; S \; , \; l \;)$	$\rightarrow$	$(\; \lfloor l-n \rfloor \; , \; S \; , \; l \;)$
$(\; \lfloor M \rfloor \; , \; N[\rho] : S \; , \; l \;)$	$\rightarrow$	$(\; N[\rho] \; , \; \lfloor M \rfloor : S \; , \; l \;)$
$(\; \lfloor B \rfloor \; , \; \lambda\!\!\lambda : S \; , \; l \;)$	$\rightarrow$	$(\; \lfloor \lambda.B \rfloor \; , \; S \; , \; l-1 \;)$
$(\; \lfloor N \rfloor \; , \; \lfloor M \rfloor : S \; , \; l \;)$	$\rightarrow$	$(\; \lfloor M\,N \rfloor \; , \; S \; , \; l \;)$
$(\; \lfloor T \rfloor \; , \; \epsilon \; , \; 0 \;)$	$\rightarrow$	T

Example with Optimised Machine

$$(\ (\lambda.0)[\overline{1}]\ ,\ 0[\overline{1}] : \lfloor 0 \rfloor : \lambda\!\!\lambda\ ,\ 1\ )$$

T	$\to$	$(\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ )$
$(\ (n+1)[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ n[\rho]\ ,\ S\ ,\ l\ )$
$(\ n[\epsilon]\ ,\ S\ ,\ l\ )$	$\to$	$(\ \lfloor n + l \rfloor\ ,\ S\ ,\ l\ )$
$(\ 0[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ \mathsf{C}\ ,\ S\ ,\ l\ )$
$(\ (M\ N)[\rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ )$	$\to$	$(\ B[\overline{N[\rho']} : \rho]\ ,\ S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ )$
$(\ \overline{n}\ ,\ S\ ,\ l\ )$	$\to$	$(\ \lfloor l - n \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor M \rfloor\ ,\ N[\rho] : S\ ,\ l\ )$	$\to$	$(\ N[\rho]\ ,\ \lfloor M \rfloor : S\ ,\ l\ )$
$(\ \lfloor B \rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ )$	$\to$	$(\ \lfloor \lambda.B \rfloor\ ,\ S\ ,\ l-1\ )$
$(\ \lfloor N \rfloor\ ,\ \lfloor M \rfloor : S\ ,\ l\ )$	$\to$	$(\ \lfloor M\ N \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor T \rfloor\ ,\ \epsilon\ ,\ 0\ )$	$\to$	T

Example with Optimised Machine

$$(\ (\lambda.0)[\overline{1}] \ , \ 0[\overline{1}] : \lfloor 0 \rfloor : \lambda\!\!\lambda \ , \ 1 \ )$$

$$
\begin{array}{|rcl|}
\hline
T & \to & (\ T[\epsilon] \ , \ \epsilon \ , \ 0 \ ) \\
\hline
(\ (n+1)[\mathsf{C} : \rho] \ , \ S \ , \ l \ ) & \to & (\ n[\rho] \ , \ S \ , \ l \ ) \\
(\ n[\epsilon] \ , \ S \ , \ l \ ) & \to & (\ \lfloor n + l \rfloor \ , \ S \ , \ l \ ) \\
(\ 0[\mathsf{C} : \rho] \ , \ S \ , \ l \ ) & \to & (\ \mathsf{C} \ , \ S \ , \ l \ ) \\
(\ (M \ N)[\rho] \ , \ S \ , \ l \ ) & \to & (\ M[\rho] \ , \ N[\rho] : S \ , \ l \ ) \\
(\ (\lambda.B)[\rho] \ , \ N[\rho'] : S \ , \ l \ ) & \to & (\ B[N[\rho'] : \rho] \ , \ S \ , \ l \ ) \\
(\ (\lambda.B)[\rho] \ , \ S \ , \ l \ ) & \to & (\ B[\overline{l+1} : \rho] \ , \ \lambda\!\!\lambda : S \ , \ l+1 \ ) \\
(\ \overline{n} \ , \ S \ , \ l \ ) & \to & (\ \lfloor l - n \rfloor \ , \ S \ , \ l \ ) \\
(\ \lfloor M \rfloor \ , \ N[\rho] : S \ , \ l \ ) & \to & (\ N[\rho] \ , \ \lfloor M \rfloor : S \ , \ l \ ) \\
(\ \lfloor B \rfloor \ , \ \lambda\!\!\lambda : S \ , \ l \ ) & \to & (\ \lfloor \lambda.B \rfloor \ , \ S \ , \ l-1 \ ) \\
(\ \lfloor N \rfloor \ , \ \lfloor M \rfloor : S \ , \ l \ ) & \to & (\ \lfloor M \ N \rfloor \ , \ S \ , \ l \ ) \\
\hline
(\ \lfloor T \rfloor \ , \ \epsilon \ , \ 0 \ ) & \to & T \\
\hline
\end{array}
$$

Example with Optimised Machine

$$(\ 0[0[\overline{1}] : \overline{1}] \ , \ \lfloor 0 \rfloor : \lambda\!\!\lambda \ , \ 1 \ )$$

T	$\rightarrow$	$(\ T[\epsilon] \ , \ \epsilon \ , \ 0 \ )$
$(\ (n+1)[\mathsf{C} : \rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ n[\rho] \ , \ S \ , \ l \ )$
$(\ n[\epsilon] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ \lfloor n + l \rfloor \ , \ S \ , \ l \ )$
$(\ 0[\mathsf{C} : \rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ \mathsf{C} \ , \ S \ , \ l \ )$
$(\ (M\,N)[\rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ M[\rho] \ , \ N[\rho] : S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ N[\rho'] : S \ , \ l \ )$	$\rightarrow$	$(\ B[N[\rho'] : \rho] \ , \ S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ B[\overline{l+1} : \rho] \ , \ \lambda\!\!\lambda : S \ , \ l+1 \ )$
$(\ \overline{n} \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ \lfloor l - n \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor M \rfloor \ , \ N[\rho] : S \ , \ l \ )$	$\rightarrow$	$(\ N[\rho] \ , \ \lfloor M \rfloor : S \ , \ l \ )$
$(\ \lfloor B \rfloor \ , \ \lambda\!\!\lambda : S \ , \ l \ )$	$\rightarrow$	$(\ \lfloor \lambda.B \rfloor \ , \ S \ , \ l-1 \ )$
$(\ \lfloor N \rfloor \ , \ \lfloor M \rfloor : S \ , \ l \ )$	$\rightarrow$	$(\ \lfloor M\,N \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor T \rfloor \ , \ \epsilon \ , \ 0 \ )$	$\rightarrow$	T

Example with Optimised Machine

$$(\ 0[0[\overline{1}] : \overline{1}] \ , \ \lfloor 0 \rfloor : \lambda\!\!\lambda \ , \ 1 \ )$$

T	$\to$	$(\ T[\epsilon] \ , \ \epsilon \ , \ 0 \ )$
$(\ (n+1)[\mathsf{C} : \rho] \ , \ S \ , \ l \ )$	$\to$	$(\ n[\rho] \ , \ S \ , \ l \ )$
$(\ n[\epsilon] \ , \ S \ , \ l \ )$	$\to$	$(\ \lfloor n + l \rfloor \ , \ S \ , \ l \ )$
$(\ 0[\mathsf{C} : \rho] \ , \ S \ , \ l \ )$	$\to$	$(\ \mathsf{C} \ , \ S \ , \ l \ )$
$(\ (M \ N)[\rho] \ , \ S \ , \ l \ )$	$\to$	$(\ M[\rho] \ , \ N[\rho] : S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ N[\rho'] : S \ , \ l \ )$	$\to$	$(\ B[\overline{N[\rho']} : \rho] \ , \ S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ S \ , \ l \ )$	$\to$	$(\ B[\overline{l+1} : \rho] \ , \ \lambda\!\!\lambda : S \ , \ l+1 \ )$
$(\ \overline{n} \ , \ S \ , \ l \ )$	$\to$	$(\ \lfloor l - n \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor M \rfloor \ , \ N[\rho] : S \ , \ l \ )$	$\to$	$(\ N[\rho] \ , \ \lfloor M \rfloor : S \ , \ l \ )$
$(\ \lfloor B \rfloor \ , \ \lambda\!\!\lambda : S \ , \ l \ )$	$\to$	$(\ \lfloor \lambda.B \rfloor \ , \ S \ , \ l - 1 \ )$
$(\ \lfloor N \rfloor \ , \ \lfloor M \rfloor : S \ , \ l \ )$	$\to$	$(\ \lfloor M \ N \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor T \rfloor \ , \ \epsilon \ , \ 0 \ )$	$\to$	T

Example with Optimised Machine

$$(\ 0[\overline{1}] \ , \ \lfloor 0 \rfloor : \lambda\!\!\lambda \ , \ 1 \ )$$

T	$\to$	$(\ T[\epsilon] \ , \ \epsilon \ , \ 0 \ )$
$(\ (n+1)[\mathsf{C} : \rho] \ , \ S \ , \ l \ )$	$\to$	$(\ n[\rho] \ , \ S \ , \ l \ )$
$(\ n[\epsilon] \ , \ S \ , \ l \ )$	$\to$	$(\ \lfloor n + l \rfloor \ , \ S \ , \ l \ )$
$(\ 0[\mathsf{C} : \rho] \ , \ S \ , \ l \ )$	$\to$	$(\ \mathsf{C} \ , \ S \ , \ l \ )$
$(\ (M\,N)[\rho] \ , \ S \ , \ l \ )$	$\to$	$(\ M[\rho] \ , \ N[\rho] : S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ N[\rho'] : S \ , \ l \ )$	$\to$	$(\ B[N[\rho'] : \rho] \ , \ S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ S \ , \ l \ )$	$\to$	$(\ B[\overline{l+1} : \rho] \ , \ \lambda\!\!\lambda : S \ , \ l+1 \ )$
$(\ \overline{n} \ , \ S \ , \ l \ )$	$\to$	$(\ \lfloor l - n \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor M \rfloor \ , \ N[\rho] : S \ , \ l \ )$	$\to$	$(\ N[\rho] \ , \ \lfloor M \rfloor : S \ , \ l \ )$
$(\ \lfloor B \rfloor \ , \ \lambda\!\!\lambda : S \ , \ l \ )$	$\to$	$(\ \lfloor \lambda.B \rfloor \ , \ S \ , \ l-1 \ )$
$(\ \lfloor N \rfloor \ , \ \lfloor M \rfloor : S \ , \ l \ )$	$\to$	$(\ \lfloor M\,N \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor T \rfloor \ , \ \epsilon \ , \ 0 \ )$	$\to$	T

Example with Optimised Machine

$$(\ 0[\overline{1}]\ ,\ \lfloor 0 \rfloor : \lambda\!\!\lambda\ ,\ 1\ )$$

$$
\begin{array}{|rcl|}
\hline
T & \to & (\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ ) \\
\hline
(\ (n+1)[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ ) & \to & (\ n[\rho]\ ,\ S\ ,\ l\ ) \\
(\ n[\epsilon]\ ,\ S\ ,\ l\ ) & \to & (\ \lfloor n+l \rfloor\ ,\ S\ ,\ l\ ) \\
(\ 0[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ ) & \to & (\ \mathsf{C}\ ,\ S\ ,\ l\ ) \\
(\ (M\ N)[\rho]\ ,\ S\ ,\ l\ ) & \to & (\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ ) \\
(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ ) & \to & (\ B[\overline{N[\rho']} : \rho]\ ,\ S\ ,\ l\ ) \\
(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ ) & \to & (\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ ) \\
(\ \overline{n}\ ,\ S\ ,\ l\ ) & \to & (\ \lfloor l-n \rfloor\ ,\ S\ ,\ l\ ) \\
(\ \lfloor M \rfloor\ ,\ N[\rho] : S\ ,\ l\ ) & \to & (\ N[\rho]\ ,\ \lfloor M \rfloor : S\ ,\ l\ ) \\
(\ \lfloor B \rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ ) & \to & (\ \lfloor \lambda.B \rfloor\ ,\ S\ ,\ l-1\ ) \\
(\ \lfloor N \rfloor\ ,\ \lfloor M \rfloor : S\ ,\ l\ ) & \to & (\ \lfloor M\ N \rfloor\ ,\ S\ ,\ l\ ) \\
\hline
(\ \lfloor T \rfloor\ ,\ \epsilon\ ,\ 0\ ) & \to & T \\
\hline
\end{array}
$$

Example with Optimised Machine

$$(\overline{1} \;,\; \lfloor 0 \rfloor : \lambda\!\!\lambda \;,\; 1)$$

T	$\to$	$(T[\epsilon] , \epsilon , 0)$
$((n+1)[\mathrm{C} : \rho] , S , l)$	$\to$	$(n[\rho] , S , l)$
$(n[\epsilon] , S , l)$	$\to$	$(\lfloor n+l \rfloor , S , l)$
$(0[\mathrm{C} : \rho] , S , l)$	$\to$	(C , S , l)
$((M\,N)[\rho] , S , l)$	$\to$	$(M[\rho] , N[\rho] : S , l)$
$((\lambda.B)[\rho] , N[\rho'] : S , l)$	$\to$	$(B[\overline{N[\rho']} : \rho] , S , l)$
$((\lambda.B)[\rho] , S , l)$	$\to$	$(B[\overline{l+1} : \rho] , \lambda\!\!\lambda : S , l+1)$
$(\overline{n} , S , l)$	$\to$	$(\lfloor l-n \rfloor , S , l)$
$(\lfloor M \rfloor , N[\rho] : S , l)$	$\to$	$(N[\rho] , \lfloor M \rfloor : S , l)$
$(\lfloor B \rfloor , \lambda\!\!\lambda : S , l)$	$\to$	$(\lfloor \lambda.B \rfloor , S , l-1)$
$(\lfloor N \rfloor , \lfloor M \rfloor : S , l)$	$\to$	$(\lfloor M\,N \rfloor , S , l)$
$(\lfloor T \rfloor , \epsilon , 0)$	$\to$	T

Example with Optimised Machine

$$(\overline{1} \ , \ \lfloor 0 \rfloor : \lambda\!\!\lambda \ , \ 1 \ )$$

T	$\to$	$(\ T[\epsilon] \ , \ \epsilon \ , \ 0 \ )$
$(\ (n+1)[\mathrm{C} : \rho] \ , \ S \ , \ l \ )$	$\to$	$(\ n[\rho] \ , \ S \ , \ l \ )$
$(\ n[\epsilon] \ , \ S \ , \ l \ )$	$\to$	$(\ \lfloor n+l \rfloor \ , \ S \ , \ l \ )$
$(\ 0[\mathrm{C} : \rho] \ , \ S \ , \ l \ )$	$\to$	$(\ \mathrm{C} \ , \ S \ , \ l \ )$
$(\ (M\,N)[\rho] \ , \ S \ , \ l \ )$	$\to$	$(\ M[\rho] \ , \ N[\rho] : S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ N[\rho'] : S \ , \ l \ )$	$\to$	$(\ B[N[\rho'] : \rho] \ , \ S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ S \ , \ l \ )$	$\to$	$(\ B[\overline{l+1} : \rho] \ , \ \lambda\!\!\lambda : S \ , \ l+1 \ )$
$(\ \overline{n} \ , \ S \ , \ l \ )$	$\to$	$(\ \lfloor l-n \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor M \rfloor \ , \ N[\rho] : S \ , \ l \ )$	$\to$	$(\ N[\rho] \ , \ \lfloor M \rfloor : S \ , \ l \ )$
$(\ \lfloor B \rfloor \ , \ \lambda\!\!\lambda : S \ , \ l \ )$	$\to$	$(\ \lfloor \lambda.B \rfloor \ , \ S \ , \ l-1 \ )$
$(\ \lfloor N \rfloor \ , \ \lfloor M \rfloor : S \ , \ l \ )$	$\to$	$(\ \lfloor M\,N \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor T \rfloor \ , \ \epsilon \ , \ 0 \ )$	$\to$	T

Example with Optimised Machine

$$(\lfloor 0 \rfloor \, , \, \lfloor 0 \rfloor : \lambda\!\!\lambda \, , \, 1)$$

T	$\to$	$(T[\epsilon] \, , \, \epsilon \, , \, 0)$
$((n+1)[\mathsf{C} : \rho] \, , \, S \, , \, l)$	$\to$	$(n[\rho] \, , \, S \, , \, l)$
$(n[\epsilon] \, , \, S \, , \, l)$	$\to$	$(\lfloor n + l \rfloor \, , \, S \, , \, l)$
$(0[\mathsf{C} : \rho] \, , \, S \, , \, l)$	$\to$	$(\mathsf{C} \, , \, S \, , \, l)$
$((M\,N)[\rho] \, , \, S \, , \, l)$	$\to$	$(M[\rho] \, , \, N[\rho] : S \, , \, l)$
$((\lambda.B)[\rho] \, , \, N[\rho'] : S \, , \, l)$	$\to$	$(B[N[\rho'] : \rho] \, , \, S \, , \, l)$
$((\lambda.B)[\rho] \, , \, S \, , \, l)$	$\to$	$(B[\overline{l+1} : \rho] \, , \, \lambda\!\!\lambda : S \, , \, l+1)$
$(\overline{n} \, , \, S \, , \, l)$	$\to$	$(\lfloor l - n \rfloor \, , \, S \, , \, l)$
$(\lfloor M \rfloor \, , \, N[\rho] : S \, , \, l)$	$\to$	$(N[\rho] \, , \, \lfloor M \rfloor : S \, , \, l)$
$(\lfloor B \rfloor \, , \, \lambda\!\!\lambda : S \, , \, l)$	$\to$	$(\lfloor \lambda.B \rfloor \, , \, S \, , \, l-1)$
$(\lfloor N \rfloor \, , \, \lfloor M \rfloor : S \, , \, l)$	$\to$	$(\lfloor M\,N \rfloor \, , \, S \, , \, l)$
$(\lfloor T \rfloor \, , \, \epsilon \, , \, 0)$	$\to$	T

Example with Optimised Machine

$$(\lfloor 0 \rfloor \,,\, \lfloor 0 \rfloor : \lambda\!\!\lambda \,,\, 1)$$

T	$\rightarrow$	$(T[\epsilon] \,,\, \epsilon \,,\, 0)$
$((n+1)[\mathsf{C} : \rho] \,,\, S \,,\, l)$	$\rightarrow$	$(n[\rho] \,,\, S \,,\, l)$
$(n[\epsilon] \,,\, S \,,\, l)$	$\rightarrow$	$(\lfloor n+l \rfloor \,,\, S \,,\, l)$
$(0[\mathsf{C} : \rho] \,,\, S \,,\, l)$	$\rightarrow$	$(\mathsf{C} \,,\, S \,,\, l)$
$((M\,N)[\rho] \,,\, S \,,\, l)$	$\rightarrow$	$(M[\rho] \,,\, N[\rho] : S \,,\, l)$
$((\lambda.B)[\rho] \,,\, N[\rho'] : S \,,\, l)$	$\rightarrow$	$(B[\overline{N[\rho']} : \rho] \,,\, S \,,\, l)$
$((\lambda.B)[\rho] \,,\, S \,,\, l)$	$\rightarrow$	$(B[\overline{l+1} : \rho] \,,\, \lambda\!\!\lambda : S \,,\, l+1)$
$(\overline{n} \,,\, S \,,\, l)$	$\rightarrow$	$(\lfloor l-n \rfloor \,,\, S \,,\, l)$
$(\lfloor M \rfloor \,,\, N[\rho] : S \,,\, l)$	$\rightarrow$	$(N[\rho] \,,\, \lfloor M \rfloor : S \,,\, l)$
$(\lfloor B \rfloor \,,\, \lambda\!\!\lambda : S \,,\, l)$	$\rightarrow$	$(\lfloor \lambda.B \rfloor \,,\, S \,,\, l-1)$
$(\lfloor N \rfloor \,,\, \lfloor M \rfloor : S \,,\, l)$	$\rightarrow$	$(\lfloor M\,N \rfloor \,,\, S \,,\, l)$
$(\lfloor T \rfloor \,,\, \epsilon \,,\, 0)$	$\rightarrow$	T

Example with Optimised Machine

$$(\lfloor 0\,0 \rfloor \,,\, \lambda\!\!\lambda \,,\, 1)$$

T	$\rightarrow$	$(T[\epsilon] \,,\, \epsilon \,,\, 0)$
$((n+1)[\mathsf{C} : \rho] \,,\, S \,,\, l)$	$\rightarrow$	$(n[\rho] \,,\, S \,,\, l)$
$(n[\epsilon] \,,\, S \,,\, l)$	$\rightarrow$	$(\lfloor n+l \rfloor \,,\, S \,,\, l)$
$(0[\mathsf{C} : \rho] \,,\, S \,,\, l)$	$\rightarrow$	$(\mathsf{C} \,,\, S \,,\, l)$
$((M\,N)[\rho] \,,\, S \,,\, l)$	$\rightarrow$	$(M[\rho] \,,\, N[\rho] : S \,,\, l)$
$((\lambda.B)[\rho] \,,\, N[\rho'] : S \,,\, l)$	$\rightarrow$	$(B[N[\rho'] : \rho] \,,\, S \,,\, l)$
$((\lambda.B)[\rho] \,,\, S \,,\, l)$	$\rightarrow$	$(B[\overline{l+1} : \rho] \,,\, \lambda\!\!\lambda : S \,,\, l+1)$
$(\overline{n} \,,\, S \,,\, l)$	$\rightarrow$	$(\lfloor l-n \rfloor \,,\, S \,,\, l)$
$(\lfloor M \rfloor \,,\, N[\rho] : S \,,\, l)$	$\rightarrow$	$(N[\rho] \,,\, \lfloor M \rfloor : S \,,\, l)$
$(\lfloor B \rfloor \,,\, \lambda\!\!\lambda : S \,,\, l)$	$\rightarrow$	$(\lfloor \lambda.B \rfloor \,,\, S \,,\, l-1)$
$(\lfloor N \rfloor \,,\, \lfloor M \rfloor : S \,,\, l)$	$\rightarrow$	$(\lfloor M\,N \rfloor \,,\, S \,,\, l)$
$(\lfloor T \rfloor \,,\, \epsilon \,,\, 0)$	$\rightarrow$	T

Example with Optimised Machine

$$(\lfloor 0\,0 \rfloor \,,\ \lambda\!\!\lambda \,,\ 1)$$

T	$\to$	$(T[\epsilon] , \epsilon , 0)$
$((n+1)[\mathrm{C} : \rho] , S , l)$	$\to$	$(n[\rho] , S , l)$
$(n[\epsilon] , S , l)$	$\to$	$(\lfloor n+l \rfloor , S , l)$
$(0[\mathrm{C} : \rho] , S , l)$	$\to$	(C , S , l)
$((M\,N)[\rho] , S , l)$	$\to$	$(M[\rho] , N[\rho] : S , l)$
$((\lambda.B)[\rho] , N[\rho'] : S , l)$	$\to$	$(B[\overline{N[\rho']} : \rho] , S , l)$
$((\lambda.B)[\rho] , S , l)$	$\to$	$(B[\overline{l+1} : \rho] , \lambda\!\!\lambda : S , l+1)$
$(\overline{n} , S , l)$	$\to$	$(\lfloor l-n \rfloor , S , l)$
$(\lfloor M \rfloor , N[\rho] : S , l)$	$\to$	$(N[\rho] , \lfloor M \rfloor : S , l)$
$(\lfloor B \rfloor , \lambda\!\!\lambda : S , l)$	$\to$	$(\lfloor \lambda.B \rfloor , S , l-1)$
$(\lfloor N \rfloor , \lfloor M \rfloor : S , l)$	$\to$	$(\lfloor M\,N \rfloor , S , l)$
$(\lfloor T \rfloor , \epsilon , 0)$	$\to$	T

Example with Optimised Machine

$$(\lfloor \lambda.0\,0 \rfloor \,,\, \epsilon \,,\, 0)$$

T	$\to$	$(T[\epsilon] \,,\, \epsilon \,,\, 0)$
$((n+1)[\mathsf{C} : \rho] \,,\, S \,,\, I)$	$\to$	$(n[\rho] \,,\, S \,,\, I)$
$(n[\epsilon] \,,\, S \,,\, I)$	$\to$	$(\lfloor n+I \rfloor \,,\, S \,,\, I)$
$(0[\mathsf{C} : \rho] \,,\, S \,,\, I)$	$\to$	$(\mathsf{C} \,,\, S \,,\, I)$
$((M\,N)[\rho] \,,\, S \,,\, I)$	$\to$	$(M[\rho] \,,\, N[\rho] : S \,,\, I)$
$((\lambda.B)[\rho] \,,\, N[\rho'] : S \,,\, I)$	$\to$	$(B[N[\rho'] : \rho] \,,\, S \,,\, I)$
$((\lambda.B)[\rho] \,,\, S \,,\, I)$	$\to$	$(B[\overline{I+1} : \rho] \,,\, \lambda\!\!\lambda : S \,,\, I+1)$
$(\overline{n} \,,\, S \,,\, I)$	$\to$	$(\lfloor I-n \rfloor \,,\, S \,,\, I)$
$(\lfloor M \rfloor \,,\, N[\rho] : S \,,\, I)$	$\to$	$(N[\rho] \,,\, \lfloor M \rfloor : S \,,\, I)$
$(\lfloor B \rfloor \,,\, \lambda\!\!\lambda : S \,,\, I)$	$\to$	$(\lfloor \lambda.B \rfloor \,,\, S \,,\, I-1)$
$(\lfloor N \rfloor \,,\, \lfloor M \rfloor : S \,,\, I)$	$\to$	$(\lfloor M\,N \rfloor \,,\, S \,,\, I)$
$(\lfloor T \rfloor \,,\, \epsilon \,,\, 0)$	$\to$	T

Example with Optimised Machine

$$(\lfloor \lambda.0\,0 \rfloor \ , \ \epsilon \ , \ 0)$$

T	$\rightarrow$	$(\ T[\epsilon] \ , \ \epsilon \ , \ 0 \ )$
$(\ (n+1)[\mathsf{C} : \rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ n[\rho] \ , \ S \ , \ l \ )$
$(\ n[\epsilon] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ \lfloor n+l \rfloor \ , \ S \ , \ l \ )$
$(\ 0[\mathsf{C} : \rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ \mathsf{C} \ , \ S \ , \ l \ )$
$(\ (M\,N)[\rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ M[\rho] \ , \ N[\rho] : S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ N[\rho'] : S \ , \ l \ )$	$\rightarrow$	$(\ B[N[\rho'] : \rho] \ , \ S \ , \ l \ )$
$(\ (\lambda.B)[\rho] \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ B[\overline{l+1} : \rho] \ , \ \lambda\!\!\lambda : S \ , \ l+1 \ )$
$(\ \overline{n} \ , \ S \ , \ l \ )$	$\rightarrow$	$(\ \lfloor l-n \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor M \rfloor \ , \ N[\rho] : S \ , \ l \ )$	$\rightarrow$	$(\ N[\rho] \ , \ \lfloor M \rfloor : S \ , \ l \ )$
$(\ \lfloor B \rfloor \ , \ \lambda\!\!\lambda : S \ , \ l \ )$	$\rightarrow$	$(\ \lfloor \lambda.B \rfloor \ , \ S \ , \ l-1 \ )$
$(\ \lfloor N \rfloor \ , \ \lfloor M \rfloor : S \ , \ l \ )$	$\rightarrow$	$(\ \lfloor M\,N \rfloor \ , \ S \ , \ l \ )$
$(\ \lfloor T \rfloor \ , \ \epsilon \ , \ 0 \ )$	$\rightarrow$	T

Example with Optimised Machine

$$\lambda.0\,0$$

T	$\to$	$(\ T[\epsilon]\ ,\ \epsilon\ ,\ 0\ )$
$(\ (n+1)[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ n[\rho]\ ,\ S\ ,\ l\ )$
$(\ n[\epsilon]\ ,\ S\ ,\ l\ )$	$\to$	$(\ \lfloor n + l \rfloor\ ,\ S\ ,\ l\ )$
$(\ 0[\mathsf{C} : \rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ \mathsf{C}\ ,\ S\ ,\ l\ )$
$(\ (M\ N)[\rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ M[\rho]\ ,\ N[\rho] : S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ N[\rho'] : S\ ,\ l\ )$	$\to$	$(\ B[N[\rho'] : \rho]\ ,\ S\ ,\ l\ )$
$(\ (\lambda.B)[\rho]\ ,\ S\ ,\ l\ )$	$\to$	$(\ B[\overline{l+1} : \rho]\ ,\ \lambda\!\!\lambda : S\ ,\ l+1\ )$
$(\ \overline{n}\ ,\ S\ ,\ l\ )$	$\to$	$(\ \lfloor l - n \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor M \rfloor\ ,\ N[\rho] : S\ ,\ l\ )$	$\to$	$(\ N[\rho]\ ,\ \lfloor M \rfloor : S\ ,\ l\ )$
$(\ \lfloor B \rfloor\ ,\ \lambda\!\!\lambda : S\ ,\ l\ )$	$\to$	$(\ \lfloor \lambda.B \rfloor\ ,\ S\ ,\ l-1\ )$
$(\ \lfloor N \rfloor\ ,\ \lfloor M \rfloor : S\ ,\ l\ )$	$\to$	$(\ \lfloor M\ N \rfloor\ ,\ S\ ,\ l\ )$
$(\ \lfloor T \rfloor\ ,\ \epsilon\ ,\ 0\ )$	$\to$	T

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