A Syntactic and Functional Correspondence between Reduction Semantics and Reduction-Free *Full* Normalisers

Álvaro García[1,2] Pablo Nogueira[2]

[1]IMDEA Software Institute

[2]Babel Research Group, Universidad Politécnica de Madrid

PEPM 2013, Rome

Interderivation

Search function $\overset{\text{decomp.}}{\Longrightarrow}$ Reduction semantics $\overset{\text{syntactic}}{\Longrightarrow}$ Abstract machine $\overset{\text{functional}}{\Longleftrightarrow}$ Reduction-free normaliser

Higher-order functional languages

Pure untyped lambda calculus:

$$\Lambda \ ::= \ x \mid (\lambda x.\Lambda) \mid (\Lambda\,\Lambda)$$

Higher-order functional languages

Pure untyped lambda calculus:

$$\Lambda \;::=\; x \mid (\lambda x.\Lambda) \mid (\Lambda\,\Lambda)$$

$$(\lambda x.B)N \rightarrow_\beta [N/x]B$$

Pure untyped lambda calculus:

$$\Lambda \ ::= \ x \mid (\lambda x.\Lambda) \mid (\Lambda\,\Lambda)$$

$$(\lambda x.B)N \rightarrow_\beta [N/x]B$$

Reduction:

$$(\lambda x.\lambda y.x)((\lambda x.x)z)$$

Higher-order functional languages

Pure untyped lambda calculus:

$$\Lambda \ ::= \ x \mid (\lambda x.\Lambda) \mid (\Lambda\,\Lambda)$$

$$(\lambda x.B)N \rightarrow_\beta [N/x]B$$

Reduction:

$$(\lambda x.\lambda y.x)(\underline{(\lambda x.x)z})$$

Pure untyped lambda calculus:

$$\Lambda \ ::= \ x \mid (\lambda x.\Lambda) \mid (\Lambda\,\Lambda)$$

$$(\lambda x.B)N \rightarrow_\beta [N/x]B$$

Reduction:

$$\underline{(\lambda x.\lambda y.x)(\underline{(\lambda x.x)z})}$$

Pure untyped lambda calculus:

$$\Lambda \ ::= \ x \mid (\lambda x.\Lambda) \mid (\Lambda\,\Lambda)$$

$$(\lambda x.B)N \rightarrow_\beta [N/x]B$$

Reduction:

$$\underline{(\lambda x.\lambda y.x)((\lambda x.x)z)}$$

Pure untyped lambda calculus:

$$\Lambda \ ::= \ x \mid (\lambda x.\Lambda) \mid (\Lambda\,\Lambda)$$

$$(\lambda x.B)N \to_\beta [N/x]B$$

Reduction:

$$\underline{(\lambda x.\lambda y.x)((\lambda x.x)z)} \to_\beta \lambda y.(\lambda x.x)z$$

Higher-order functional languages

Pure untyped lambda calculus:

$$\Lambda \ ::= \ x \mid (\lambda x.\Lambda) \mid (\Lambda\,\Lambda)$$

$$(\lambda x.B)N \rightarrow_\beta [N/x]B$$

Reduction:

$$\underline{(\lambda x.\lambda y.x)((\lambda x.x)z)} \rightarrow_\beta \lambda y.(\lambda x.x)z$$

$$\begin{array}{lcl} \text{WHNF} & ::= & \lambda x.\Lambda \\ & \mid & x\,\{\Lambda\}^* \end{array}$$

Higher-order functional languages

Pure untyped lambda calculus:

$$\Lambda \;::=\; x \mid (\lambda x.\Lambda) \mid (\Lambda\,\Lambda)$$

$$(\lambda x.B)N \rightarrow_\beta [N/x]B$$

Reduction:

$$\underline{(\lambda x.\lambda y.x)((\lambda x.x)z)} \rightarrow_\beta \lambda y.\underline{(\lambda x.x)z}$$

$$\begin{array}{lcl} \text{WHNF} & ::= & \lambda x.\Lambda \\ & \mid & x\,\{\Lambda\}^* \end{array}$$

Higher-order functional languages

Pure untyped lambda calculus:

$$\Lambda \ ::= \ x \mid (\lambda x.\Lambda) \mid (\Lambda\,\Lambda)$$

$$(\lambda x.B)N \rightarrow_\beta [N/x]B$$

Reduction:

$$\underline{(\lambda x.\lambda y.x)((\lambda x.x)z)} \rightarrow_\beta \lambda y.\underline{(\lambda x.x)z} \rightarrow_\beta \lambda y.z$$

$$\begin{array}{lcl} \text{WHNF} & ::= & \lambda x.\Lambda \\ & \mid & x\,\{\Lambda\}^* \end{array}$$

Higher-order functional languages

Pure untyped lambda calculus:

$$\Lambda \;::=\; x \mid (\lambda x.\Lambda) \mid (\Lambda\,\Lambda)$$

$$(\lambda x.B)N \to_\beta [N/x]B$$

Reduction:

$$\underline{(\lambda x.\lambda y.x)((\lambda x.x)z)} \to_\beta \lambda y.\underline{(\lambda x.x)z} \to_\beta \lambda y.z$$

$$\begin{array}{rcl} \mathrm{WHNF} & ::= & \lambda x.\Lambda \\ & \mid & x\,\{\Lambda\}^* \end{array} \qquad \begin{array}{rcl} \mathrm{NF} & ::= & \lambda x.\mathrm{NF} \\ & \mid & x\,\{\mathrm{NF}\}^* \end{array}$$

Existing techniques

$$\frac{}{(\lambda x.B)N \rightarrow_{bn} [N/x]B}\ (\beta) \qquad \frac{M \rightarrow_{bn} M'}{M\,N \rightarrow_{bn} M'\,N}\ (\mu)$$

$$\frac{}{(\lambda x.B)N \to_{bn} [N/x]B}\ (\beta) \qquad \frac{M \to_{bn} M'}{M\,N \to_{bn} M'\,N}\ (\mu)$$

$\Downarrow$ decomposition

$$\frac{}{(\lambda x.B)N \rightarrow_{bn} [N/x]B}\ (\beta) \qquad \frac{M \rightarrow_{bn} M'}{M\,N \rightarrow_{bn} M'\,N}\ (\mu)$$

$\Downarrow$ (CPS, defunc, ...)

$$\frac{}{(\lambda x.B)N \to_{bn} [N/x]B}\ (\beta) \qquad \frac{M \to_{bn} M'}{M\,N \to_{bn} M'\,N}\ (\mu)$$

$\Downarrow$ (CPS, defunc, ...)

Contexts: $\mathsf{C}_{bn}[\,] \ ::= \ [\,] \mid \mathsf{C}_{bn}[\,]\,\Lambda$

β-rule: $\mathsf{C}_{bn}[(\lambda x.B)N] \to_{\beta} \mathsf{C}_{bn}[[N/x]B]$

$$\frac{}{(\lambda x.B)N \to_{bn} [N/x]B}\ (\beta) \qquad \frac{M \to_{bn} M'}{M\,N \to_{bn} M'\,N}\ (\mu)$$

$\Downarrow$ (CPS, defunc, ...) redex at the root

Contexts: $C_{bn}[\,] ::= [\,] \mid C_{bn}[\,]\Lambda$

β-rule: $C_{bn}[(\lambda x.B)N] \to_\beta C_{bn}[[N/x]B]$

$$\frac{}{(\lambda x.B)N \rightarrow_{bn} [N/x]B}\ (\beta) \qquad \frac{M \rightarrow_{bn} M'}{M\,N \rightarrow_{bn} M'\,N}\ (\mu)$$

$\Downarrow$ (CPS, defunc, ...) recursing over the operator

Contexts: $\mathsf{C}_{bn}[\,] ::= [\,] \mid \mathsf{C}_{bn}[\,]\Lambda$

β-rule: $\mathsf{C}_{bn}[(\lambda x.B)N] \rightarrow_\beta \mathsf{C}_{bn}[[N/x]B]$

$$\frac{}{(\lambda x.B)N \to_{bn} [N/x]B}\ (\beta) \qquad \frac{M \to_{bn} M'}{M\,N \to_{bn} M'\,N}\ (\mu)$$

$\Downarrow$ (CPS, defunc, ...)

Contexts: $\mathsf{C}_{bn}[\,] ::= [\,] \mid \mathsf{C}_{bn}[\,]\Lambda$

β-rule: $\mathsf{C}_{bn}[(\lambda x.B)N] \to_\beta \mathsf{C}_{bn}[[N/x]B]$

$\Downarrow$ syntactic

$$\frac{}{(\lambda x.B)N \to_{bn} [N/x]B}\ (\beta) \qquad \frac{M \to_{bn} M'}{M\,N \to_{bn} M'\,N}\ (\mu)$$

$\Downarrow$ (CPS, defunc, ...)

Contexts: $\mathsf{C}_{bn}[\,] ::= [\,] \mid \mathsf{C}_{bn}[\,]\Lambda$

β-rule: $\mathsf{C}_{bn}[(\lambda x.B)N] \to_\beta \mathsf{C}_{bn}[[N/x]B]$

$\Downarrow$ (refocus, LWF, inline)

$$\frac{}{(\lambda x.B)N \rightarrow_{bn} [N/x]B}\ (\beta) \qquad\qquad \frac{M \rightarrow_{bn} M'}{M\,N \rightarrow_{bn} M'\,N}\ (\mu)$$

$\Downarrow$ (CPS, defunc, ...)

Contexts: $\mathsf{C}_{bn}[\,] \;::=\; [\,] \mid \mathsf{C}_{bn}[\,]\,\Lambda$

β-rule: $\mathsf{C}_{bn}[(\lambda x.B)N] \rightarrow_{\beta} \mathsf{C}_{bn}[[N/x]B]$

$\Downarrow$ (refocus, LWF, inline)

(1)	T	$\rightarrow$	$(T, \mathbf{e}, \mathsf{C}_0)$
(2)	$(x, \mathbf{e}, S)$	$\rightarrow$	$(x, \mathbf{c}, S)$
(3)	$(\lambda x.B, \mathbf{e}, S)$	$\rightarrow$	$(\lambda x.B, \mathbf{c}, S)$
(4)	$(M\,N, \mathbf{e}, S)$	$\rightarrow$	$(M, \mathbf{e}, \mathsf{C}_1(N) : S)$
(5)	$(\lambda x.B, \mathbf{c}, \mathsf{C}_1(N) : S)$	$\rightarrow$	$([N/x]B, \mathbf{e}, S)$
(6)	$(M \not\equiv \lambda x.B, \mathbf{c}, \mathsf{C}_1(N) : S)$	$\rightarrow$	$(M\,N, \mathbf{c}, S)$
(7)	$(T, \mathbf{c}, \mathsf{C}_0)$	$\rightarrow$	T

$$\frac{}{(\lambda x.B)N \to_{bn} [N/x]B} \; (\beta) \qquad \frac{M \to_{bn} M'}{M\,N \to_{bn} M'\,N} \; (\mu)$$

$\Downarrow$ (CPS, defunc, ...)

Contexts: $C_{bn}[\,] ::= [\,] \mid C_{bn}[\,]\Lambda$

β-rule: $C_{bn}[(\lambda x.B)N] \to_{\beta} C_{bn}[[N/x]B]$

$\Downarrow$ (refocus, LWF, inline) init

(1)	T	$\to$	$(T, \mathbf{e}, C_0)$
(2)	$(x, \mathbf{e}, S)$	$\to$	$(x, \mathbf{c}, S)$
(3)	$(\lambda x.B, \mathbf{e}, S)$	$\to$	$(\lambda x.B, \mathbf{c}, S)$
(4)	$(M\,N, \mathbf{e}, S)$	$\to$	$(M, \mathbf{e}, C_1(N) : S)$
(5)	$(\lambda x.B, \mathbf{c}, C_1(N) : S)$	$\to$	$([N/x]B, \mathbf{e}, S)$
(6)	$(M \not\equiv \lambda x.B, \mathbf{c}, C_1(N) : S)$	$\to$	$(M\,N, \mathbf{c}, S)$
(7)	$(T, \mathbf{c}, C_0)$	$\to$	T

$$\frac{}{(\lambda x.B)N \to_{bn} [N/x]B}\ (\beta) \qquad \frac{M \to_{bn} M'}{M\,N \to_{bn} M'\,N}\ (\mu)$$

$\Downarrow$ (CPS, defunc, ...)

Contexts: $\mathsf{C}_{bn}[\,] \ ::= \ [\,] \mid \mathsf{C}_{bn}[\,]\Lambda$

β-rule: $\mathsf{C}_{bn}[(\lambda x.B)N] \to_\beta \mathsf{C}_{bn}[[N/x]B]$

$\Downarrow$ (refocus, LWF, inline)

empty

(1)	T	$\to$	$(T, \mathbf{e}, \mathsf{C}_0)$
(2)	$(x, \mathbf{e}, S)$	$\to$	$(x, \mathbf{c}, S)$
(3)	$(\lambda x.B, \mathbf{e}, S)$	$\to$	$(\lambda x.B, \mathbf{c}, S)$
(4)	$(M\,N, \mathbf{e}, S)$	$\to$	$(M, \mathbf{e}, \mathsf{C}_1(N) : S)$
(5)	$(\lambda x.B, \mathbf{c}, \mathsf{C}_1(N) : S)$	$\to$	$([N/x]B, \mathbf{e}, S)$
(6)	$(M \not\equiv \lambda x.B, \mathbf{c}, \mathsf{C}_1(N) : S)$	$\to$	$(M\,N, \mathbf{c}, S)$
(7)	$(T, \mathbf{c}, \mathsf{C}_0)$	$\to$	T

$$\frac{}{(\lambda x.B)N \rightarrow_{bn} [N/x]B}\ (\beta) \qquad \frac{M \rightarrow_{bn} M'}{M\ N \rightarrow_{bn} M'\ N}\ (\mu)$$

$\Downarrow$ (CPS, defunc, ...)

Contexts: $C_{bn}[\,] ::= [\,] \mid C_{bn}[\,]\Lambda$

β-rule: $C_{bn}[(\lambda x.B)N] \rightarrow_\beta C_{bn}[[N/x]B]$

$\Downarrow$ (refocus, LWF, inline) operator

(1)	T	$\rightarrow$	$(T, \mathbf{e}, C_0)$
(2)	$(x, \mathbf{e}, S)$	$\rightarrow$	$(x, \mathbf{c}, S)$
(3)	$(\lambda x.B, \mathbf{e}, S)$	$\rightarrow$	$(\lambda x.B, \mathbf{c}, S)$
(4)	$(M\ N, \mathbf{e}, S)$	$\rightarrow$	$(M, \mathbf{e}, C_1(N) : S)$
(5)	$(\lambda x.B, \mathbf{c}, C_1(N) : S)$	$\rightarrow$	$([N/x]B, \mathbf{e}, S)$
(6)	$(M \not\equiv \lambda x.B, \mathbf{c}, C_1(N) : S)$	$\rightarrow$	$(M\ N, \mathbf{c}, S)$
(7)	$(T, \mathbf{c}, C_0)$	$\rightarrow$	T

$$\frac{}{(\lambda x.B)N \to_{bn} [N/x]B}\ (\beta) \qquad \frac{M \to_{bn} M'}{M\,N \to_{bn} M'\,N}\ (\mu)$$

$\Downarrow$ (CPS, defunc, ...)

Contexts: $\mathsf{C}_{bn}[\,] \ ::= \ [\,] \mid \mathsf{C}_{bn}[\,]\,\Lambda$

β-rule: $\mathsf{C}_{bn}[(\lambda x.B)N] \to_\beta \mathsf{C}_{bn}[[N/x]B]$

$\Downarrow$ (refocus, LWF, inline) eval

(1)	T	$\to$	$(T, \mathbf{e}, \mathsf{C}_0)$
(2)	$(x, \mathbf{e}, S)$	$\to$	$(x, \mathbf{c}, S)$
(3)	$(\lambda x.B, \mathbf{e}, S)$	$\to$	$(\lambda x.B, \mathbf{c}, S)$
(4)	$(M\,N, \mathbf{e}, S)$	$\to$	$(M, \mathbf{e}, \mathsf{C}_1(N) : S)$
(5)	$(\lambda x.B, \mathbf{c}, \mathsf{C}_1(N) : S)$	$\to$	$([N/x]B, \mathbf{e}, S)$
(6)	$(M \not\equiv \lambda x.B, \mathbf{c}, \mathsf{C}_1(N) : S)$	$\to$	$(M\,N, \mathbf{c}, S)$
(7)	$(T, \mathbf{c}, \mathsf{C}_0)$	$\to$	T

$$\frac{}{(\lambda x.B)N \to_{bn} [N/x]B}\ (\beta) \qquad \frac{M \to_{bn} M'}{M\ N \to_{bn} M'\ N}\ (\mu)$$

$\Downarrow$ (CPS, defunc, ...)

Contexts: $\mathsf{C}_{bn}[\,] ::= [\,] \mid \mathsf{C}_{bn}[\,]\Lambda$

β-rule: $\mathsf{C}_{bn}[(\lambda x.B)N] \to_\beta \mathsf{C}_{bn}[[N/x]B]$

$\Downarrow$ (refocus, LWF, inline) apply cont

(1)	T	$\to$	$(T, \mathbf{e}, \mathsf{C}_0)$
(2)	$(x, \mathbf{e}, S)$	$\to$	$(x, \mathbf{c}, S)$
(3)	$(\lambda x.B, \mathbf{e}, S)$	$\to$	$(\lambda x.B, \mathbf{c}, S)$
(4)	$(M\,N, \mathbf{e}, S)$	$\to$	$(M, \mathbf{e}, \mathsf{C}_1(N) : S)$
(5)	$(\lambda x.B, \mathbf{c}, \mathsf{C}_1(N) : S)$	$\to$	$([N/x]B, \mathbf{e}, S)$
(6)	$(M \not\equiv \lambda x.B, \mathbf{c}, \mathsf{C}_1(N) : S)$	$\to$	$(M\,N, \mathbf{c}, S)$
(7)	$(T, \mathbf{c}, \mathsf{C}_0)$	$\to$	T

$$\frac{}{(\lambda x.B)N \rightarrow_{bn} [N/x]B}\ (\beta) \qquad\qquad \frac{M \rightarrow_{bn} M'}{M\,N \rightarrow_{bn} M'\,N}\ (\mu)$$

$\Downarrow$ (CPS, defunc, ...)

Contexts: $C_{bn}[\,] \ ::= \ [\,] \mid C_{bn}[\,]\,\Lambda$

β-rule: $C_{bn}[(\lambda x.B)N] \rightarrow_\beta C_{bn}[[N/x]B]$

$\Downarrow$ (refocus, LWF, inline)

(1)	T	$\rightarrow$	$(T, \mathbf{e}, C_0)$
(2)	$(x, \mathbf{e}, S)$	$\rightarrow$	$(x, \mathbf{c}, S)$
(3)	$(\lambda x.B, \mathbf{e}, S)$	$\rightarrow$	$(\lambda x.B, \mathbf{c}, S)$
(4)	$(M\,N, \mathbf{e}, S)$	$\rightarrow$	$(M, \mathbf{e}, C_1(N) : S)$
(5)	$(\lambda x.B, \mathbf{c}, C_1(N) : S)$	$\rightarrow$	$([N/x]B, \mathbf{e}, S)$
(6)	$(M \not\equiv \lambda x.B, \mathbf{c}, C_1(N) : S)$	$\rightarrow$	$(M\,N, \mathbf{c}, S)$
(7)	$(T, \mathbf{c}, C_0)$	$\rightarrow$	T

$\Updownarrow$ functional

$$\frac{}{(\lambda x.B)N \to_{bn} [N/x]B}\ (\beta) \qquad \frac{M \to_{bn} M'}{M\,N \to_{bn} M'\,N}\ (\mu)$$

$\Downarrow$ (CPS, defunc, ...)

Contexts: $\mathsf{C}_{bn}[\,] \ ::= \ [\,] \mid \mathsf{C}_{bn}[\,]\,\Lambda$

β-rule: $\mathsf{C}_{bn}[(\lambda x.B)N] \to_\beta \mathsf{C}_{bn}[[N/x]B]$

$\Downarrow$ (refocus, LWF, inline)

(1)	T	$\to$	$(T, \mathbf{e}, \mathsf{C}_0)$
(2)	$(x, \mathbf{e}, S)$	$\to$	$(x, \mathbf{c}, S)$
(3)	$(\lambda x.B, \mathbf{e}, S)$	$\to$	$(\lambda x.B, \mathbf{c}, S)$
(4)	$(M\,N, \mathbf{e}, S)$	$\to$	$(M, \mathbf{e}, \mathsf{C}_1(N) : S)$
(5)	$(\lambda x.B, \mathbf{c}, \mathsf{C}_1(N) : S)$	$\to$	$([N/x]B, \mathbf{e}, S)$
(6)	$(M \not\equiv \lambda x.B, \mathbf{c}, \mathsf{C}_1(N) : S)$	$\to$	$(M\,N, \mathbf{c}, S)$
(7)	$(T, \mathbf{c}, \mathsf{C}_0)$	$\to$	T

$\Updownarrow$ (refunc, CPS^{-1})

$$\frac{}{(\lambda x.B)N \to_{bn} [N/x]B}\ (\beta) \qquad \frac{M \to_{bn} M'}{M\,N \to_{bn} M'\,N}\ (\mu)$$

$\Downarrow$ (CPS, defunc, ...)

Contexts: $\mathsf{C}_{bn}[\,] ::= [\,] \mid \mathsf{C}_{bn}[\,]\,\Lambda$

β-rule: $\mathsf{C}_{bn}[(\lambda x.B)N] \to_\beta \mathsf{C}_{bn}[[N/x]B]$

$\Downarrow$ (refocus, LWF, inline)

(1)	T	$\to$	$(T, \mathbf{e}, \mathsf{C}_0)$
(2)	$(x, \mathbf{e}, S)$	$\to$	$(x, \mathbf{c}, S)$
(3)	$(\lambda x.B, \mathbf{e}, S)$	$\to$	$(\lambda x.B, \mathbf{c}, S)$
(4)	$(M\,N, \mathbf{e}, S)$	$\to$	$(M, \mathbf{e}, \mathsf{C}_1(N) : S)$
(5)	$(\lambda x.B, \mathbf{c}, \mathsf{C}_1(N) : S)$	$\to$	$([N/x]B, \mathbf{e}, S)$
(6)	$(M \not\equiv \lambda x.B, \mathbf{c}, \mathsf{C}_1(N) : S)$	$\to$	$(M\,N, \mathbf{c}, S)$
(7)	$(T, \mathbf{c}, \mathsf{C}_0)$	$\to$	T

$\Updownarrow$ (refunc, CPS^{-1})

$$\frac{}{x \Downarrow_{bn} x}\ (\textsc{Bn-Var}) \qquad \frac{}{\lambda x.B \Downarrow_{bn} \lambda x.B}\ (\textsc{Bn-Abs})$$

$$\frac{M \Downarrow_{bn} M' \quad M' \equiv \lambda x.B \quad [N/x]B \Downarrow_{bn} B'}{M\,N \Downarrow_{bn} B'}\ (\textsc{Bn-Con})$$

$$\frac{M \Downarrow_{bn} M' \quad M' \not\equiv \lambda x.B}{M\,N \Downarrow_{bn} M'\,N}\ (\textsc{Bn-Neu})$$

$$\frac{}{(\lambda x.B)N \rightarrow_{bn} [N/x]B}\ (\beta) \qquad \frac{M \rightarrow_{bn} M'}{M\,N \rightarrow_{bn} M'\,N}\ (\mu)$$

$\Downarrow$ (CPS, defunc, ...)

Contexts: $\mathsf{C}_{bn}[\,] ::= [\,] \mid \mathsf{C}_{bn}[\,]\,\Lambda$

β-rule: $\mathsf{C}_{bn}[(\lambda x.B)N] \rightarrow_\beta \mathsf{C}_{bn}[[N/x]B]$

$\Downarrow$ (refocus, LWF, inline)

(1)	T	$\rightarrow$	$(T, \mathbf{e}, \mathsf{C}_0)$
(2)	$(x, \mathbf{e}, S)$	$\rightarrow$	$(x, \mathbf{c}, S)$
(3)	$(\lambda x.B, \mathbf{e}, S)$	$\rightarrow$	$(\lambda x.B, \mathbf{c}, S)$
(4)	$(M\,N, \mathbf{e}, S)$	$\rightarrow$	$(M, \mathbf{e}, \mathsf{C}_1(N) : S)$
(5)	$(\lambda x.B, \mathbf{c}, \mathsf{C}_1(N) : S)$	$\rightarrow$	$([N/x]B, \mathbf{e}, S)$
(6)	$(M \not\equiv \lambda x.B, \mathbf{c}, \mathsf{C}_1(N) : S)$	$\rightarrow$	$(M\,N, \mathbf{c}, S)$
(7)	$(T, \mathbf{c}, \mathsf{C}_0)$	$\rightarrow$	T

$\Updownarrow$ (refunc, CPS^{-1}) irred

$$\frac{}{x \Downarrow_{bn} x}\ (\textsc{Bn-Var}) \qquad \frac{}{\lambda x.B \Downarrow_{bn} \lambda x.B}\ (\textsc{Bn-Abs})$$

$$\frac{M \Downarrow_{bn} M' \quad M' \equiv \lambda x.B \quad [N/x]B \Downarrow_{bn} B'}{M\,N \Downarrow_{bn} B'}\ (\textsc{Bn-Con})$$

$$\frac{M \Downarrow_{bn} M' \quad M' \not\equiv \lambda x.B}{M\,N \Downarrow_{bn} M'\,N}\ (\textsc{Bn-Neu})$$

$$\frac{}{(\lambda x.B)N \to_{bn} [N/x]B}\ (\beta) \qquad \frac{M \to_{bn} M'}{M\,N \to_{bn} M'\,N}\ (\mu)$$

$\Downarrow$ (CPS, defunc, ...)

Contexts: $\mathsf{C}_{bn}[\,] \ ::= \ [\,] \mid \mathsf{C}_{bn}[\,]\,\Lambda$

β-rule: $\mathsf{C}_{bn}[(\lambda x.B)N] \to_\beta \mathsf{C}_{bn}[[N/x]B]$

$\Downarrow$ (refocus, LWF, inline)

(1)	T	$\to$	$(T, \mathbf{e}, \mathsf{C}_0)$
(2)	$(x, \mathbf{e}, S)$	$\to$	$(x, \mathbf{c}, S)$
(3)	$(\lambda x.B, \mathbf{e}, S)$	$\to$	$(\lambda x.B, \mathbf{c}, S)$
(4)	$(M\,N, \mathbf{e}, S)$	$\to$	$(M, \mathbf{e}, \mathsf{C}_1(N) : S)$
(5)	$(\lambda x.B, \mathbf{c}, \mathsf{C}_1(N) : S)$	$\to$	$([N/x]B, \mathbf{e}, S)$
(6)	$(M \not\equiv \lambda x.B, \mathbf{c}, \mathsf{C}_1(N) : S)$	$\to$	$(M\,N, \mathbf{c}, S)$
(7)	$(T, \mathbf{c}, \mathsf{C}_0)$	$\to$	T

$\Updownarrow$ (refunc, CPS^{-1}) contract

$$\frac{}{x \Downarrow_{bn} x}\ (\textsc{Bn-Var}) \qquad \frac{}{\lambda x.B \Downarrow_{bn} \lambda x.B}\ (\textsc{Bn-Abs})$$

$$\frac{M \Downarrow_{bn} M' \quad M' \equiv \lambda x.B \quad [N/x]B \Downarrow_{bn} B'}{M\,N \Downarrow_{bn} B'}\ (\textsc{Bn-Con})$$

$$\frac{M \Downarrow_{bn} M' \quad M' \not\equiv \lambda x.B}{M\,N \Downarrow_{bn} M'\,N}\ (\textsc{Bn-Neu})$$

$$\frac{}{(\lambda x.B)N \rightarrow_{bn} [N/x]B}\ (\beta) \qquad \frac{M \rightarrow_{bn} M'}{M\,N \rightarrow_{bn} M'\,N}\ (\mu)$$

$\Downarrow$ (CPS, defunc, ...)

Contexts: $C_{bn}[\,] ::= [\,] \mid C_{bn}[\,]\,\Lambda$

β-rule: $C_{bn}[(\lambda x.B)N] \rightarrow_\beta C_{bn}[[N/x]B]$

$\Downarrow$ (refocus, LWF, inline)

(1)	T	$\rightarrow$	$(T, \mathbf{e}, C_0)$
(2)	$(x, \mathbf{e}, S)$	$\rightarrow$	$(x, \mathbf{c}, S)$
(3)	$(\lambda x.B, \mathbf{e}, S)$	$\rightarrow$	$(\lambda x.B, \mathbf{c}, S)$
(4)	$(M\,N, \mathbf{e}, S)$	$\rightarrow$	$(M, \mathbf{e}, C_1(N) : S)$
(5)	$(\lambda x.B, \mathbf{c}, C_1(N) : S)$	$\rightarrow$	$([N/x]B, \mathbf{e}, S)$
(6)	$(M \not\equiv \lambda x.B, \mathbf{c}, C_1(N) : S)$	$\rightarrow$	$(M\,N, \mathbf{c}, S)$
(7)	$(T, \mathbf{c}, C_0)$	$\rightarrow$	T

$\Updownarrow$ (refunc, CPS^{-1}) neutral

$$\frac{}{x \Downarrow_{bn} x}\ (\text{Bn-Var}) \qquad \frac{}{\lambda x.B \Downarrow_{bn} \lambda x.B}\ (\text{Bn-Abs})$$

$$\frac{M \Downarrow_{bn} M' \quad M' \equiv \lambda x.B \quad [N/x]B \Downarrow_{bn} B'}{M\,N \Downarrow_{bn} B'}\ (\text{Bn-Con})$$

$$\frac{M \Downarrow_{bn} M' \quad M' \not\equiv \lambda x.B}{M\,N \Downarrow_{bn} M'\,N}\ (\text{Bn-Neu})$$

$$\frac{}{(\lambda x.B)N \to_{bn} [N/x]B}\ (\beta) \qquad \frac{M \to_{bn} M'}{M\,N \to_{bn} M'\,N}\ (\mu)$$

$\Downarrow$ (CPS, defunc, ...)

Contexts: $\mathsf{C}_{bn}[\,] \ ::= \ [\,] \mid \mathsf{C}_{bn}[\,]\,\Lambda$

β-rule: $\mathsf{C}_{bn}[(\lambda x.B)N] \to_\beta \mathsf{C}_{bn}[[N/x]B]$

$\Downarrow$ (refocus, LWF, inline)

(1)	T	$\to$	$(T, \mathbf{e}, \mathsf{C}_0)$
(2)	$(x, \mathbf{e}, S)$	$\to$	$(x, \mathbf{c}, S)$
(3)	$(\lambda x.B, \mathbf{e}, S)$	$\to$	$(\lambda x.B, \mathbf{c}, S)$
(4)	$(M\,N, \mathbf{e}, S)$	$\to$	$(M, \mathbf{e}, \mathsf{C}_1(N) : S)$
(5)	$(\lambda x.B, \mathbf{c}, \mathsf{C}_1(N) : S)$	$\to$	$([N/x]B, \mathbf{e}, S)$
(6)	$(M \not\equiv \lambda x.B, \mathbf{c}, \mathsf{C}_1(N) : S)$	$\to$	$(M\,N, \mathbf{c}, S)$
(7)	$(T, \mathbf{c}, \mathsf{C}_0)$	$\to$	T

$\Updownarrow$ (refunc, CPS^{-1})

$$\frac{}{x \Downarrow_{bn} x}\ (\textsc{Bn-Var}) \qquad \frac{}{\lambda x.B \Downarrow_{bn} \lambda x.B}\ (\textsc{Bn-Abs})$$

$$\frac{M \Downarrow_{bn} M' \quad M' \equiv \lambda x.B \quad [N/x]B \Downarrow_{bn} B'}{M\,N \Downarrow_{bn} B'}\ (\textsc{Bn-Con})$$

$$\frac{M \Downarrow_{bn} M' \quad M' \not\equiv \lambda x.B}{M\,N \Downarrow_{bn} M'\,N}\ (\textsc{Bn-Neu})$$

However...

Normal order is hybrid

$$\frac{}{x \Downarrow_{bn} x}\ (\textsc{Bn-Var}) \qquad \frac{}{\lambda x.B \Downarrow_{bn} \lambda x.B}\ (\textsc{Bn-Abs})$$

$$\frac{M \Downarrow_{bn} M' \quad M' \equiv \lambda x.B \quad [N/x]B \Downarrow_{bn} B'}{M\ N \Downarrow_{bn} B'}\ (\textsc{Bn-Con})$$

$$\frac{M \Downarrow_{bn} M' \quad M' \not\equiv \lambda x.B}{M\ N \Downarrow_{bn} M'\ N}\ (\textsc{Bn-Neu})$$

$$\frac{}{x \Downarrow_{no} x}\ (\textsc{No-Var}) \qquad \frac{B \Downarrow_{no} B'}{\lambda x.B \Downarrow_{no} \lambda x.B'}\ (\textsc{No-Abs})$$

$$\frac{M \Downarrow_{bn} M' \quad M' \equiv \lambda x.B \quad [N/x]B \Downarrow_{no} B'}{M\ N \Downarrow_{no} B'}\ (\textsc{No-Con})$$

$$\frac{M \Downarrow_{bn} M' \quad M' \not\equiv \lambda x.B \quad M' \Downarrow_{no} M'' \quad N \Downarrow_{no} N'}{M\ N \Downarrow_{no} M''\ N'}\ (\textsc{No-Neu})$$

Normal order is hybrid

$$\frac{}{x \Downarrow_{bn} x}\ (\textsc{Bn-Var}) \qquad \frac{}{\lambda x.B \Downarrow_{bn} \lambda x.B}\ (\textsc{Bn-Abs})$$

$$\frac{M \Downarrow_{bn} M' \quad M' \equiv \lambda x.B \quad [N/x]B \Downarrow_{bn} B'}{M\ N \Downarrow_{bn} B'}\ (\textsc{Bn-Con})$$

$$\frac{M \Downarrow_{bn} M' \quad M' \not\equiv \lambda x.B}{M\ N \Downarrow_{bn} M'\ N}\ (\textsc{Bn-Neu})$$

$$\frac{}{x \Downarrow_{no} x}\ (\textsc{No-Var}) \qquad \frac{B \Downarrow_{no} B'}{\lambda x.B \Downarrow_{no} \lambda x.B'}\ (\textsc{No-Abs})$$

$$\frac{M \Downarrow_{bn} M' \quad M' \equiv \lambda x.B \quad [N/x]B \Downarrow_{no} B'}{M\ N \Downarrow_{no} B'}\ (\textsc{No-Con})$$

$$\frac{M \Downarrow_{bn} M' \quad M' \not\equiv \lambda x.B \quad M' \Downarrow_{no} M'' \quad N \Downarrow_{no} N'}{M\ N \Downarrow_{no} M''\ N'}\ (\textsc{No-Neu})$$

Our contribution

Search function

$$\frac{}{(\lambda x.B)N \rightarrow_{no} [N/x]B}\ (\beta) \qquad \frac{M \notin \text{WHNF} \quad M \rightarrow_{no} M'}{M\,N \rightarrow_{no} M'\,N}\ (\mu 1)$$

$$\frac{M \in \text{WHNF} \quad M \not\equiv \lambda x.B \quad M \rightarrow_{no} M'}{M\,N \rightarrow_{no} M'\,N}\ (\mu 2)$$

$$\frac{M \in \text{NF} \quad M \not\equiv \lambda x.B \quad N \rightarrow_{no} N'}{M\,N \rightarrow_{no} M\,N'}\ (\nu)$$

$$\frac{B \rightarrow_{no} B'}{\lambda x.B \rightarrow_{no} \lambda x.B'}\ (\xi)$$

Search function

$$\frac{}{(\lambda x.B)N \to_{no} [N/x]B}\ (\beta) \qquad \frac{M \notin \text{WHNF} \quad M \to_{no} M'}{M\, N \to_{no} M'\, N}\ (\mu 1)$$

$$\frac{M \in \text{WHNF} \quad M \not\equiv \lambda x.B \quad M \to_{no} M'}{M\, N \to_{no} M'\, N}\ (\mu 2)$$

$$\frac{M \in \text{NF} \quad M \not\equiv \lambda x.B \quad N \to_{no} N'}{M\, N \to_{no} M\, N'}\ (\nu)$$

$$\frac{B \to_{no} B'}{\lambda x.B \to_{no} \lambda x.B'}\ (\xi)$$

Reduction semantics

Neutrals in nf:	NNF	::=	$x \mid \text{NNF}\,\text{NF}$
Contexts:	$\text{C}_{no}[\,]$	::=	$[\,] \mid \text{C}_{bn}[\,]\,\Lambda \mid \lambda x.\text{C}_{no}[\,] \mid \text{C}_{ne}[\,]$
	$\text{C}_{bn}[\,]$	::=	$[\,] \mid \text{C}_{bn}[\,]\,\Lambda$
	$\text{C}_{ne}[\,]$	::=	$\text{NNF}\,\text{C}_{no}[\,] \mid \text{C}_{ne}[\,]\,\Lambda$
β-rule:	$\text{C}_{no}[(\lambda x.B)N] \rightarrow_\beta \text{C}_{no}[[N/x]B]$		

Reduction semantics

Neutrals in nf:	NNF	$::=$	$x \mid$ NNF NF
Contexts:	$\mathsf{C}_{no}[\ ]$	$::=$	$[\ ] \mid \mathsf{C}_{bn}[\ ]\,\Lambda \mid \lambda x.\mathsf{C}_{no}[\ ] \mid \mathsf{C}_{ne}[\ ]$
	$\mathsf{C}_{bn}[\ ]$	$::=$	$[\ ] \mid \mathsf{C}_{bn}[\ ]\,\Lambda$
	$\mathsf{C}_{ne}[\ ]$	$::=$	NNF $\mathsf{C}_{no}[\ ] \mid \mathsf{C}_{ne}[\ ]\,\Lambda$
β-rule:	$\mathsf{C}_{no}[(\lambda x.B)N] \rightarrow_\beta \mathsf{C}_{no}[[N/x]B]$		

From contexts to defunctionalised continuations

$$
\begin{array}{lcl}
\mathsf{C}_{no}[\,] & ::= & [\,] \mid \mathsf{C}_{bn}[\,]\,\Lambda \mid \lambda x.\mathsf{C}_{no}[\,] \mid \mathsf{C}_{ne}[\,] \\
\mathsf{C}_{bn}[\,] & ::= & [\,] \mid \mathsf{C}_{bn}[\,]\,\Lambda \\
\mathsf{C}_{ne}[\,] & ::= & \mathsf{NNF}\,\mathsf{C}_{no}[\,] \mid \mathsf{C}_{ne}[\,]\,\Lambda
\end{array}
$$

From contexts to defunctionalised continuations

$$
\begin{array}{rcl}
C_{no}[\,] & ::= & [\,] \mid C_{bn}[\,]\,\Lambda \mid \lambda x.C_{no}[\,] \mid C_{ne}[\,] \\
C_{bn}[\,] & ::= & [\,] \mid C_{bn}[\,]\,\Lambda \\
C_{ne}[\,] & ::= & \mathrm{NNF}\; C_{no}[\,] \mid C_{ne}[\,]\,\Lambda
\end{array}
$$

$$
\begin{array}{rcl}
C_{no}[\,] & \equiv & C_0 \\
C_{no}[\,] \Rightarrow \lambda x.C_{no}[\,] & \equiv & C_2(x) \\
C_{no}[\,] \Rightarrow C_{bn}[\,]\,\Lambda & \equiv & C_3(\Lambda) \\
C_{bn}[\,] \Rightarrow C_{bn}[\,]\,\Lambda & \equiv & C_1(\Lambda) \\
C_{bn}[\,] \Rightarrow [\,] & \equiv & \text{empty} \\
C_{no}[\,] \Rightarrow C_{ne}[\,] & \equiv & \text{empty} \\
C_{ne}[\,] \Rightarrow C_{ne}[\,]\,\Lambda & \equiv & C_4(\Lambda) \\
C_{ne}[\,] \Rightarrow \mathrm{NNF}\; C_{no}[\,] & \equiv & C_5(\mathrm{NNF}) \\
C_{no}[\,] \Rightarrow [\,] & \equiv & \text{empty}
\end{array}
$$

From contexts to defunctionalised continuations

$$
\begin{array}{rcl}
C_{no}[\,] & ::= & \{\lambda x.\}^* \, [\,] \, \{\Lambda\}^* \, \Lambda \\
 & | & \{\lambda x.\}^* \, [\,] \\
 & | & \{\lambda x.\}^* \, \mathrm{NNF} \, C_{no}[\,] \, \{\Lambda\}^*
\end{array}
$$

$$
\begin{array}{rcl}
C_{no}[\,] & \equiv & C_0 \\
C_{no}[\,] \Rightarrow \lambda x.C_{no}[\,] & \equiv & C_2(x) \\
C_{no}[\,] \Rightarrow C_{bn}[\,]\,\Lambda & \equiv & C_3(\Lambda) \\
C_{bn}[\,] \Rightarrow C_{bn}[\,]\,\Lambda & \equiv & C_1(\Lambda) \\
C_{bn}[\,] \Rightarrow [\,] & \equiv & \text{empty} \\
C_{no}[\,] \Rightarrow C_{ne}[\,] & \equiv & \text{empty} \\
C_{ne}[\,] \Rightarrow C_{ne}[\,]\,\Lambda & \equiv & C_4(\Lambda) \\
C_{ne}[\,] \Rightarrow \mathrm{NNF}\,C_{no}[\,] & \equiv & C_5(\mathrm{NNF}) \\
C_{no}[\,] \Rightarrow [\,] & \equiv & \text{empty}
\end{array}
$$

$$
\begin{array}{rcl}
C_{no}[\,] & ::= & \{\lambda x.\}^*\,[\,]\,\{\Lambda\}^*\,\Lambda \\
& | & \{\lambda x.\}^*\,[\,] \\
& | & \{\lambda x.\}^*\,\mathrm{NNF}\,C_{no}[\,]\,\{\Lambda\}^*
\end{array}
$$

$$
\begin{array}{rcl}
C_{no}[\,] & \equiv & C_0 \\
C_{no}[\,] \Rightarrow \lambda x.C_{no}[\,] & \equiv & C_2(x) \\
C_{no}[\,] \Rightarrow C_{bn}[\,]\,\Lambda & \equiv & C_3(\Lambda) \\
C_{bn}[\,] \Rightarrow C_{bn}[\,]\,\Lambda & \equiv & C_1(\Lambda) \\
C_{bn}[\,] \Rightarrow [\,] & \equiv & \text{empty} \\
C_{no}[\,] \Rightarrow C_{ne}[\,] & \equiv & \text{empty} \\
C_{ne}[\,] \Rightarrow C_{ne}[\,]\,\Lambda & \equiv & C_4(\Lambda) \\
C_{ne}[\,] \Rightarrow \mathrm{NNF}\,C_{no}[\,] & \equiv & C_5(\mathrm{NNF}) \\
C_{no}[\,] \Rightarrow [\,] & \equiv & \text{empty}
\end{array}
$$

$$
\begin{array}{rcl}
\mathsf{C}_{no}[\,] & ::= & \{\lambda x.\}^{*}\,[\,]\,\{\Lambda\}^{*}\,\Lambda \\
& \mid & \{\lambda x.\}^{*}\,[\,] \\
& \mid & \{\lambda x.\}^{*}\,\mathsf{NNF}\,\mathsf{C}_{no}[\,]\,\{\Lambda\}^{*}
\end{array}
$$

$$
\begin{array}{rcl}
\mathsf{C}_{no}[\,] & \equiv & \mathsf{C}_0 \\
\mathsf{C}_{no}[\,] \Rightarrow \lambda x.\mathsf{C}_{no}[\,] & \equiv & \mathsf{C}_2(x) \\
\mathsf{C}_{no}[\,] \Rightarrow \mathsf{C}_{bn}[\,]\,\Lambda & \equiv & \mathsf{C}_3(\Lambda) \\
\mathsf{C}_{bn}[\,] \Rightarrow \mathsf{C}_{bn}[\,]\,\Lambda & \equiv & \mathsf{C}_1(\Lambda) \\
\mathsf{C}_{bn}[\,] \Rightarrow [\,] & \equiv & \mathsf{empty} \\
\mathsf{C}_{no}[\,] \Rightarrow \mathsf{C}_{ne}[\,] & \equiv & \mathsf{empty} \\
\mathsf{C}_{ne}[\,] \Rightarrow \mathsf{C}_{ne}[\,]\,\Lambda & \equiv & \mathsf{C}_4(\Lambda) \\
\mathsf{C}_{ne}[\,] \Rightarrow \mathsf{NNF}\,\mathsf{C}_{no}[\,] & \equiv & \mathsf{C}_5(\mathsf{NNF}) \\
\mathsf{C}_{no}[\,] \Rightarrow [\,] & \equiv & \mathsf{empty}
\end{array}
$$

From contexts to defunctionalised continuations

$$
\begin{array}{rcl}
\mathsf{C}_{no}[\,] & ::= & \{\lambda x.\}^* \, [\,] \, \{\Lambda\}^* \, \Lambda \\
 & | & \{\lambda x.\}^* \, [\,] \\
 & | & \{\lambda x.\}^* \, \mathsf{NNF} \, \mathsf{C}_{no}[\,] \, \{\Lambda\}^*
\end{array}
$$

$$
\begin{array}{rcl}
\mathsf{C}_{no}[\,] & \equiv & \mathsf{C}_0 \\
\mathsf{C}_{no}[\,] \Rightarrow \lambda x.\mathsf{C}_{no}[\,] & \equiv & \mathsf{C}_2(x) \\
\mathsf{C}_{no}[\,] \Rightarrow \mathsf{C}_{bn}[\,]\,\Lambda & \equiv & \mathsf{C}_3(\Lambda) \\
\mathsf{C}_{bn}[\,] \Rightarrow \mathsf{C}_{bn}[\,]\,\Lambda & \equiv & \mathsf{C}_1(\Lambda) \\
\mathsf{C}_{bn}[\,] \Rightarrow [\,] & \equiv & \mathsf{empty} \\
\mathsf{C}_{no}[\,] \Rightarrow \mathsf{C}_{ne}[\,] & \equiv & \mathsf{empty} \\
\mathsf{C}_{ne}[\,] \Rightarrow \mathsf{C}_{ne}[\,]\,\Lambda & \equiv & \mathsf{C}_4(\Lambda) \\
\mathsf{C}_{ne}[\,] \Rightarrow \mathsf{NNF}\,\mathsf{C}_{no}[\,] & \equiv & \mathsf{C}_5(\mathsf{NNF}) \\
\mathsf{C}_{no}[\,] \Rightarrow [\,] & \equiv & \mathsf{empty}
\end{array}
$$

From contexts to defunctionalised continuations

$$
\begin{array}{rcl}
\mathsf{C}_{no}[\,] & ::= & \{\lambda x.\}^* \, [\,] \, \{\Lambda\}^* \, \Lambda \\
& | & \{\lambda x.\}^* \, [\,] \\
& | & \{\lambda x.\}^* \, \mathsf{NNF} \, \mathsf{C}_{no}[\,] \, \{\Lambda\}^*
\end{array}
$$

$$
\begin{array}{rcl}
\mathsf{C}_{no}[\,] & \equiv & \mathsf{C}_0 \\
\mathsf{C}_{no}[\,] \Rightarrow \lambda x.\mathsf{C}_{no}[\,] & \equiv & \mathsf{C}_2(x) \\
\mathsf{C}_{no}[\,] \Rightarrow \mathsf{C}_{bn}[\,]\,\Lambda & \equiv & \mathsf{C}_3(\Lambda) \\
\mathsf{C}_{bn}[\,] \Rightarrow \mathsf{C}_{bn}[\,]\,\Lambda & \equiv & \mathsf{C}_1(\Lambda) \\
\mathsf{C}_{bn}[\,] \Rightarrow [\,] & \equiv & \mathsf{empty} \\
\mathsf{C}_{no}[\,] \Rightarrow \mathsf{C}_{ne}[\,] & \equiv & \mathsf{empty} \\
\mathsf{C}_{ne}[\,] \Rightarrow \mathsf{C}_{ne}[\,]\,\Lambda & \equiv & \mathsf{C}_4(\Lambda) \\
\mathsf{C}_{ne}[\,] \Rightarrow \mathsf{NNF}\,\mathsf{C}_{no}[\,] & \equiv & \mathsf{C}_5(\mathsf{NNF}) \\
\mathsf{C}_{no}[\,] \Rightarrow [\,] & \equiv & \mathsf{empty}
\end{array}
$$

From contexts to defunctionalised continuations

$$
\begin{array}{rcl}
C_{no}[\,] & ::= & \{\lambda x.\}^* \, {\color{red}[\,]} \, \{\Lambda\}^* \, \Lambda \\
& | & \{\lambda x.\}^* \, [\,] \\
& | & \{\lambda x.\}^* \, \mathrm{NNF} \, C_{no}[\,] \, \{\Lambda\}^*
\end{array}
$$

$$
\begin{array}{rcl}
C_{no}[\,] & \equiv & C_0 \\
C_{no}[\,] \Rightarrow \lambda x. C_{no}[\,] & \equiv & C_2(x) \\
C_{no}[\,] \Rightarrow C_{bn}[\,]\,\Lambda & \equiv & C_3(\Lambda) \\
C_{bn}[\,] \Rightarrow C_{bn}[\,]\,\Lambda & \equiv & C_1(\Lambda) \\
C_{bn}[\,] \Rightarrow [\,] & \equiv & {\color{red}\mathrm{empty}} \\
C_{no}[\,] \Rightarrow C_{ne}[\,] & \equiv & \mathrm{empty} \\
C_{ne}[\,] \Rightarrow C_{ne}[\,]\,\Lambda & \equiv & C_4(\Lambda) \\
C_{ne}[\,] \Rightarrow \mathrm{NNF}\, C_{no}[\,] & \equiv & C_5(\mathrm{NNF}) \\
C_{no}[\,] \Rightarrow [\,] & \equiv & \mathrm{empty}
\end{array}
$$

From contexts to defunctionalised continuations

$$
\begin{array}{rcl}
C_{no}[\,] & ::= & \{\lambda x.\}^*\,[\,]\,\{\Lambda\}^*\,\Lambda \\
& \mid & \{\lambda x.\}^*\,[\,] \\
& \mid & \{\lambda x.\}^*\,\mathrm{NNF}\,C_{no}[\,]\,\{\Lambda\}^*
\end{array}
$$

$$
\begin{array}{rcl}
C_{no}[\,] & \equiv & C_0 \\
C_{no}[\,] \Rightarrow \lambda x.C_{no}[\,] & \equiv & C_2(x) \\
C_{no}[\,] \Rightarrow C_{bn}[\,]\,\Lambda & \equiv & C_3(\Lambda) \\
C_{bn}[\,] \Rightarrow C_{bn}[\,]\,\Lambda & \equiv & C_1(\Lambda) \\
C_{bn}[\,] \Rightarrow [\,] & \equiv & \text{empty} \\
C_{no}[\,] \Rightarrow C_{ne}[\,] & \equiv & \text{empty} \\
C_{ne}[\,] \Rightarrow C_{ne}[\,]\,\Lambda & \equiv & C_4(\Lambda) \\
C_{ne}[\,] \Rightarrow \mathrm{NNF}\,C_{no}[\,] & \equiv & C_5(\mathrm{NNF}) \\
C_{no}[\,] \Rightarrow [\,] & \equiv & \text{empty}
\end{array}
$$

From contexts to defunctionalised continuations

$$
\begin{array}{rcl}
C_{no}[\,] & ::= & \{\lambda x.\}^* \, [\,] \, \{\Lambda\}^* \, \Lambda \\
& | & \{\lambda x.\}^* \, [\,] \\
& | & \{\lambda x.\}^* \, \mathsf{NNF} \, C_{no}[\,] \, \{\Lambda\}^*
\end{array}
$$

$$
\begin{array}{rcl}
C_{no}[\,] & \equiv & C_0 \\
C_{no}[\,] \Rightarrow \lambda x.C_{no}[\,] & \equiv & C_2(x) \\
C_{no}[\,] \Rightarrow C_{bn}[\,]\,\Lambda & \equiv & C_3(\Lambda) \\
C_{bn}[\,] \Rightarrow C_{bn}[\,]\,\Lambda & \equiv & C_1(\Lambda) \\
C_{bn}[\,] \Rightarrow [\,] & \equiv & \mathsf{empty} \\
C_{no}[\,] \Rightarrow C_{ne}[\,] & \equiv & \mathsf{empty} \\
C_{ne}[\,] \Rightarrow C_{ne}[\,]\,\Lambda & \equiv & C_4(\Lambda) \\
C_{ne}[\,] \Rightarrow \mathsf{NNF}\,C_{no}[\,] & \equiv & C_5(\mathsf{NNF}) \\
C_{no}[\,] \Rightarrow [\,] & \equiv & \mathsf{empty}
\end{array}
$$

From contexts to defunctionalised continuations

$$
\begin{array}{rcl}
C_{no}[\,] & ::= & \{\lambda x.\}^*\,[\,]\,\{\Lambda\}^*\,\Lambda \\
& \mid & \{\lambda x.\}^*\,[\,] \\
& \mid & \{\lambda x.\}^*\,\mathrm{NNF}\,C_{no}[\,]\,\{\Lambda\}^*
\end{array}
$$

$$
\begin{array}{rcl}
C_{no}[\,] & \equiv & C_0 \\
C_{no}[\,] \Rightarrow \lambda x.C_{no}[\,] & \equiv & C_2(x) \\
C_{no}[\,] \Rightarrow C_{bn}[\,]\,\Lambda & \equiv & C_3(\Lambda) \\
C_{bn}[\,] \Rightarrow C_{bn}[\,]\,\Lambda & \equiv & C_1(\Lambda) \\
C_{bn}[\,] \Rightarrow [\,] & \equiv & \mathrm{empty} \\
C_{no}[\,] \Rightarrow C_{ne}[\,] & \equiv & \mathrm{empty} \\
C_{ne}[\,] \Rightarrow C_{ne}[\,]\,\Lambda & \equiv & C_4(\Lambda) \\
C_{ne}[\,] \Rightarrow \mathrm{NNF}\,C_{no}[\,] & \equiv & C_5(\mathrm{NNF}) \\
C_{no}[\,] \Rightarrow [\,] & \equiv & \mathrm{empty}
\end{array}
$$

Well-formed context stack

$$
\begin{array}{lcl}
\mathsf{S} & ::= & \{\mathsf{D}\}^{?}\,\mathsf{C}_0 \\
\mathsf{D} & ::= & \{\{\mathsf{C}_1(\Lambda) :\}^{*}\,\mathsf{C}_3(\Lambda) :\}^{?}\,\{\mathsf{C}_2(x) :\}^{*} \\
 & | & \{\mathsf{D}\}^{?}\,\mathsf{C}_5(\mathsf{NNF}) : \{\mathsf{C}_4(\Lambda) :\}^{*}\,\{\mathsf{C}_2(x) :\}^{*}
\end{array}
$$

Well-formed context stack

$$
\begin{array}{rcl}
\mathsf{S} & ::= & \{\mathsf{D}\}^{?}\, \mathsf{C}_0 \\
\mathsf{D} & ::= & \{\{\mathsf{C}_1(\Lambda) :\}^{*}\, \mathsf{C}_3(\Lambda) :\}^{?}\, \{\mathsf{C}_2(x) :\}^{*} \\
 & | & \{\mathsf{D}\}^{?}\, \mathsf{C}_5(\mathsf{NNF}) : \{\mathsf{C}_4(\Lambda) :\}^{*}\, \{\mathsf{C}_2(x) :\}^{*}
\end{array}
$$

$$
\begin{array}{lcll}
[\,] & \equiv & \mathsf{C}_0 & \\
[\,]\, N & \equiv & \mathsf{C}_3(N) : \mathsf{C}_0 & \\
\lambda x.[\,] & \equiv & \mathsf{C}_2(x) : \mathsf{C}_0 & \\
[\,]\, N''\, N'\, N & \equiv & \mathsf{C}_1(N'') : \mathsf{C}_1(N') : \mathsf{C}_3(N) : \mathsf{C}_0 & \\
\lambda x.\lambda y.[\,]\, N & \equiv & \mathsf{C}_3(N) : \mathsf{C}_2(y) : \mathsf{C}_2(x) : \mathsf{C}_0 & \\
xM[\,]N & \equiv & \mathsf{C}_5(xM) : \mathsf{C}_4(N) : \mathsf{C}_0 & (xM \in \mathsf{NNF}) \\
\ldots & & &
\end{array}
$$

Example with context stacks

$[(\lambda x.y(I\,I))I\,I,\ \mathsf{C}_0]$ (remember $I \equiv \lambda x.x$)

Example with context stacks

$[(\lambda x.y(I\,I))I\,I,\ \mathsf{C}_0]$ (remember $I \equiv \lambda x.x$)

$\xrightarrow{\text{decompose\_nf}}$

$[(\lambda x.y(I\,I))I,\ \mathsf{C}_3(I) : \mathsf{C}_0]$

Example with context stacks

$[(\lambda x.y(I\,I))I\,I,\ \mathsf{C}_0]$ (remember $I \equiv \lambda x.x$)

$\xrightarrow{\text{decompose\_nf}}$

$[(\lambda x.y(I\,I))I,\ \mathsf{C}_3(I) : \mathsf{C}_0]$

$\downarrow \beta$

$[y(I\,I),\ \mathsf{C}_3(I) : \mathsf{C}_0]$

Example with context stacks

$[(\lambda x.y(I\,I))I\,I,\ \mathsf{C}_0]$ (remember $I \equiv \lambda x.x$)

decompose_nf

$[(\lambda x.y(I\,I))I,\ \mathsf{C}_3(I) : \mathsf{C}_0]$

$\downarrow \beta$

$[y(I\,I),\ \mathsf{C}_3(I) : \mathsf{C}_0]$

recompose

$[y(I\,I)I,\ \mathsf{C}_0]$

Example with context stacks

$[(\lambda x.y(I\,I))I\,I,\ \mathsf{C}_0]$ (remember $I \equiv \lambda x.x$)

decompose_nf $\searrow$

$[(\lambda x.y(I\,I))I,\ \mathsf{C}_3(I) : \mathsf{C}_0]$

$\downarrow \beta$

$[y(I\,I),\ \mathsf{C}_3(I) : \mathsf{C}_0]$

recompose $\swarrow$

$[y(I\,I)I,\ \mathsf{C}_0]$

decompose_nf $\searrow$

$[I\,I,\ \mathsf{C}_5(y) : \mathsf{C}_4(I) : \mathsf{C}_0]$

Example with context stacks

$$[(\lambda x.y(I\,I))I\,I,\ \ \mathsf{C}_0] \qquad \text{(remember } I \equiv \lambda x.x\text{)}$$

decompose_nf

$$[(\lambda x.y(I\,I))I,\ \ \mathsf{C}_3(I) : \mathsf{C}_0]$$

$$\downarrow \beta$$

$$[y(I\,I),\ \ \mathsf{C}_3(I) : \mathsf{C}_0]$$

recompose

$$[y(I\,I)I,\ \ \mathsf{C}_0]$$

decompose_nf

$$[I\,I,\ \ \mathsf{C}_5(y) : \mathsf{C}_4(I) : \mathsf{C}_0]$$

$$\downarrow \beta$$

$$[I,\ \ \mathsf{C}_5(y) : \mathsf{C}_4(I) : \mathsf{C}_0]$$

Example with context stacks

$$[(\lambda x.y(I\,I))I\,I,\ \mathsf{C}_0]$$

(remember $I \equiv \lambda x.x$)

decompose_nf

$$[(\lambda x.y(I\,I))I,\ \mathsf{C}_3(I) : \mathsf{C}_0]$$

$$\downarrow \beta$$

$$[y(I\,I),\ \mathsf{C}_3(I) : \mathsf{C}_0]$$

recompose

$$[y(I\,I)I,\ \mathsf{C}_0]$$

decompose_nf

$$[I\,I,\ \mathsf{C}_5(y) : \mathsf{C}_4(I) : \mathsf{C}_0]$$

$$\downarrow \beta$$

$$[I,\ \mathsf{C}_5(y) : \mathsf{C}_4(I) : \mathsf{C}_0]$$

recompose

$$[y\,I\,I,\ \mathsf{C}_0]$$

Example with context stacks and with refocus

$$[(\lambda x.y(I\,I))I\,I,\ \ \mathsf{C}_0]$$

(remember $I \equiv \lambda x.x$)

decompose_nf

$$[(\lambda x.y(I\,I))I,\ \ \mathsf{C}_3(I) : \mathsf{C}_0]$$

$$\downarrow \beta$$

$$[y(I\,I),\ \ \mathsf{C}_3(I) : \mathsf{C}_0]$$

refocus

$$[I\,I,\ \ \mathsf{C}_5(y) : \mathsf{C}_4(I) : \mathsf{C}_0]$$

$$\downarrow \beta$$

$$[I,\ \ \mathsf{C}_5(y) : \mathsf{C}_4(I) : \mathsf{C}_0]$$

recompose

$$[y\,I\,I,\ \ \mathsf{C}_0]$$

Example with context stacks and with refocus

$[(\lambda x.y(I\,I))I\,I,\ \mathsf{C}_0]$ (remember $I \equiv \lambda x.x$)

decompose_nf

$[(\lambda x.y(I\,I))I,\ \mathsf{C}_3(I) : \mathsf{C}_0]$

$\downarrow \beta$

$[y(I\,I),\ \mathsf{C}_3(I) : \mathsf{C}_0]$

$\downarrow$ decompose_whnf

$[\mathbf{y}(\mathbf{I}\,\mathbf{I}),\ \mathsf{C}_3(I) : \mathsf{C}_0]$

$[I\,I,\ \mathsf{C}_5(y) : \mathsf{C}_4(I) : \mathsf{C}_0]$

$\downarrow \beta$

$[I,\ \mathsf{C}_5(y) : \mathsf{C}_4(I) : \mathsf{C}_0]$

recompose

$[y\,I\,I,\ \mathsf{C}_0]$

Example with context stacks and with refocus

$[(\lambda x.y(I\,I))I\,I,\ \mathsf{C}_0]$ (remember $I \equiv \lambda x.x$)

decompose_nf

$[(\lambda x.y(I\,I))I,\ \mathsf{C}_3(I) : \mathsf{C}_0]$

$\downarrow \beta$

$[y(I\,I),\ \mathsf{C}_3(I) : \mathsf{C}_0]$

$\downarrow$ decompose_whnf

$[\mathbf{y(I\,I)},\ \mathsf{C}_3(I) : \mathsf{C}_0]$

$\downarrow$ decompose_cont

$[I\,I,\ \mathsf{C}_5(y) : \mathsf{C}_4(I) : \mathsf{C}_0]$

$\downarrow \beta$

$[I,\ \mathsf{C}_5(y) : \mathsf{C}_4(I) : \mathsf{C}_0]$

recompose

$[y\,I\,I,\ \mathsf{C}_0]$

Example with context stacks and with refocus

$$[(\lambda x.y(I\,I))I\,I,\ \mathsf{C}_0]$$

(remember $I \equiv \lambda x.x$)

decompose_nf

$$[(\lambda x.y(I\,I))I,\ \mathsf{C}_3(I) : \mathsf{C}_0]$$

$\downarrow \beta$

$$[y(I\,I),\ \mathsf{C}_3(I) : \mathsf{C}_0]$$

$\downarrow$ decompose_whnf

$$[\mathbf{y(I\,I)},\ \mathsf{C}_3(I) : \mathsf{C}_0]$$

$\downarrow$ decompose_cont

$$[I\,I,\ \mathsf{C}_5(y) : \mathsf{C}_4(I) : \mathsf{C}_0]$$

$\downarrow \beta$

$$[I,\ \mathsf{C}_5(y) : \mathsf{C}_4(I) : \mathsf{C}_0]$$

refocus

$$[y\,I\,I,\ \mathsf{C}_0]$$

Example with context stacks and with refocus

$[(\lambda x.y(I\,I))I\,I,\ \mathsf{C}_0]$ (remember $I \equiv \lambda x.x$)

$\searrow$ decompose_nf

$$[(\lambda x.y(I\,I))I,\ \mathsf{C}_3(I) : \mathsf{C}_0]$$

$\downarrow \beta$

$$[y(I\,I),\ \mathsf{C}_3(I) : \mathsf{C}_0]$$

$\downarrow$ decompose_whnf

$$[\mathbf{y}(\mathbf{I}\,\mathbf{I}),\ \mathsf{C}_3(I) : \mathsf{C}_0]$$

$\downarrow$ decompose_cont

$$[I\,I,\ \mathsf{C}_5(y) : \mathsf{C}_4(I) : \mathsf{C}_0]$$

$\downarrow \beta$

$$[I,\ \mathsf{C}_5(y) : \mathsf{C}_4(I) : \mathsf{C}_0]$$

$\downarrow$ decompose_nf

$$[\mathbf{y}\,\mathbf{I}\,\mathbf{I},\ \mathsf{C}_0]$$

$[y\,I\,I,\ \mathsf{C}_0]$

Example with context stacks and with refocus

$[(\lambda x.y(I\,I))I\,I,\ C_0]$ (remember $I \equiv \lambda x.x$)

decompose_nf ↘

$[(\lambda x.y(I\,I))I,\ C_3(I) : C_0]$

↓ β

$[y(I\,I),\ C_3(I) : C_0]$

↓ decompose_whnf

$[\mathbf{y(I\,I)},\ C_3(I) : C_0]$

↓ decompose_cont

$[I\,I,\ C_5(y) : C_4(I) : C_0]$

↓ β

$[I,\ C_5(y) : C_4(I) : C_0]$

↓ decompose_nf

$[\mathbf{y\,I\,I},\ C_0]$

↙ decompose_cont

$[y\,I\,I,\ C_0]$

(1)	T	$\rightarrow$	$(T, \mathbf{n}, \mathsf{C}_0)$
(2)	$(x, \mathbf{n}, S)$	$\rightarrow$	$(x, \mathbf{c}, S)$
(3)	$(\lambda x.B, \mathbf{n}, S)$	$\rightarrow$	$(B, \mathbf{n}, \mathsf{C}_2(x) : S)$
(4)	$(M\,N, \mathbf{n}, S)$	$\rightarrow$	$(M, \mathbf{w}, \mathsf{C}_3(N) : S)$
(5)	$(x, \mathbf{w}, S)$	$\rightarrow$	$(x, \mathbf{c}, S)$
(6)	$(\lambda x.B, \mathbf{w}, S)$	$\rightarrow$	$(\lambda x.B, \mathbf{c}, S)$
(7)	$(M\,N, \mathbf{w}, S)$	$\rightarrow$	$(M, \mathbf{w}, \mathsf{C}_1(N) : S)$
(8)	$(\lambda x.B, \mathbf{c}, \mathsf{C}_1(N) : S)$	$\rightarrow$	
	if $S \equiv \mathsf{C}_{(1\mid 3)}(\_) : S'$ then $([N/x]B, \mathbf{w}, S)$		
			else $([N/x]B, \mathbf{n}, S)$
(9)	$(M \not\equiv \lambda x.B, \mathbf{c}, \mathsf{C}_1(N) : S)$	$\rightarrow$	$(M\,N, \mathbf{c}, S)$
(10)	$(B, \mathbf{c}, \mathsf{C}_2(x) : S)$	$\rightarrow$	$(\lambda x.B, \mathbf{c}, S)$
(11)	$(\lambda x.B, \mathbf{c}, \mathsf{C}_3(N) : S)$	$\rightarrow$	
	if $S \equiv \mathsf{C}_{(1\mid 3)}(\_) : S'$ then $([N/x]B, \mathbf{w}, S)$		
			else $([N/x]B, \mathbf{n}, S)$
(12)	$(M \not\equiv \lambda x.B, \mathbf{c}, \mathsf{C}_3(N) : S)$	$\rightarrow$	$(M, \mathbf{n}, \mathsf{C}_4(N) : S)$
(13)	$(M, \mathbf{c}, \mathsf{C}_4(N) : S)$	$\rightarrow$	$(N, \mathbf{n}, \mathsf{C}_5(M) : S)$
(14)	$(N, \mathbf{c}, \mathsf{C}_5(M) : S)$	$\rightarrow$	$(M\,N, \mathbf{c}, S)$
(15)	$(T, \mathbf{c}, \mathsf{C}_0)$	$\rightarrow$	T

(1)	T	$\rightarrow$	$(T, \mathbf{n}, \mathsf{C}_0)$
(2)	$(x, \mathbf{n}, S)$	$\rightarrow$	$(x, \mathbf{c}, S)$
(3)	$(\lambda x.B, \mathbf{n}, S)$	$\rightarrow$	$(B, \mathbf{n}, \mathsf{C}_2(x) : S)$
(4)	$(M\,N, \mathbf{n}, S)$	$\rightarrow$	$(M, \mathbf{w}, \mathsf{C}_3(N) : S)$
(5)	$(x, \mathbf{w}, S)$	$\rightarrow$	$(x, \mathbf{c}, S)$
(6)	$(\lambda x.B, \mathbf{w}, S)$	$\rightarrow$	$(\lambda x.B, \mathbf{c}, S)$
(7)	$(M\,N, \mathbf{w}, S)$	$\rightarrow$	$(M, \mathbf{w}, \mathsf{C}_1(N) : S)$
(8)	$(\lambda x.B, \mathbf{c}, \mathsf{C}_1(N) : S)$	$\rightarrow$	
	if $S \equiv \mathsf{C}_{(1\|3)}(\_) : S'$ then $([N/x]B, \mathbf{w}, S)$ else $([N/x]B, \mathbf{n}, S)$		
(9)	$(M \not\equiv \lambda x.B, \mathbf{c}, \mathsf{C}_1(N) : S)$	$\rightarrow$	$(M\,N, \mathbf{c}, S)$
(10)	$(B, \mathbf{c}, \mathsf{C}_2(x) : S)$	$\rightarrow$	$(\lambda x.B, \mathbf{c}, S)$
(11)	$(\lambda x.B, \mathbf{c}, \mathsf{C}_3(N) : S)$	$\rightarrow$	
	if $S \equiv \mathsf{C}_{(1\|3)}(\_) : S'$ then $([N/x]B, \mathbf{w}, S)$ else $([N/x]B, \mathbf{n}, S)$		
(12)	$(M \not\equiv \lambda x.B, \mathbf{c}, \mathsf{C}_3(N) : S)$	$\rightarrow$	$(M, \mathbf{n}, \mathsf{C}_4(N) : S)$
(13)	$(M, \mathbf{c}, \mathsf{C}_4(N) : S)$	$\rightarrow$	$(N, \mathbf{n}, \mathsf{C}_5(M) : S)$
(14)	$(N, \mathbf{c}, \mathsf{C}_5(M) : S)$	$\rightarrow$	$(M\,N, \mathbf{c}, S)$
(15)	$(T, \mathbf{c}, \mathsf{C}_0)$	$\rightarrow$	T

Recovering the shallow-inspection property

$$
\begin{array}{rcl}
\mathsf{S} & ::= & \{\mathsf{D}\}^{?}\, \mathsf{C}_0 \\
\mathsf{D} & ::= & \{\{\mathsf{C}_1(\Lambda) :\}^{*}\, \mathsf{C}_3(\Lambda) :\}^{?}\, \{\mathsf{C}_2(x) :\}^{*} \\
 & | & \{\mathsf{D}\}^{?}\, \mathsf{C}_5(\mathsf{NNF}) : \{\mathsf{C}_4(\Lambda) :\}^{*}\, \{\mathsf{C}_2(x) :\}^{*}
\end{array}
$$

$$
\begin{array}{rcl}
\mathsf{S} & ::= & \{\mathsf{D}\}^{?}\,\mathsf{C}_0 \\
\mathsf{D} & ::= & \{\{\mathsf{C}_1(\Lambda) :\}^{*}\,\mathsf{C}_3(\Lambda) :\}^{?}\,\{\mathsf{C}_2(x) :\}^{*} \\
 & | & \{\mathsf{D}\}^{?}\,\mathsf{C}_5(\mathsf{NNF}) : \{\mathsf{C}_4(\Lambda) :\}^{*}\,\{\mathsf{C}_2(x) :\}^{*}
\end{array}
$$

If the current continuation is C_1, the next is always C_1 or C_3.
If the current continuation is C_3, the next is never C_1 nor C_3.

(1)	T	$\rightarrow$	$(T, \mathbf{n}, \mathsf{C}_0)$
(2)	$(x, \mathbf{n}, S)$	$\rightarrow$	$(x, \mathbf{c}, S)$
(3)	$(\lambda x.B, \mathbf{n}, S)$	$\rightarrow$	$(B, \mathbf{n}, \mathsf{C}_2(x) : S)$
(4)	$(M\,N, \mathbf{n}, S)$	$\rightarrow$	$(M, \mathbf{w}, \mathsf{C}_3(N) : S)$
(5)	$(x, \mathbf{w}, S)$	$\rightarrow$	$(x, \mathbf{c}, S)$
(6)	$(\lambda x.B, \mathbf{w}, S)$	$\rightarrow$	$(\lambda x.B, \mathbf{c}, S)$
(7)	$(M\,N, \mathbf{w}, S)$	$\rightarrow$	$(M, \mathbf{w}, \mathsf{C}_1(N) : S)$
(8)	$(\lambda x.B, \mathbf{c}, \mathsf{C}_1(N) : S)$	$\rightarrow$	
	if $S \equiv \mathsf{C}_{(1\|3)}(\_) : S'$ then $([N/x]B, \mathbf{w}, S)$		
			else $([N/x]B, \mathbf{n}, S)$
(9)	$(M \not\equiv \lambda x.B, \mathbf{c}, \mathsf{C}_1(N) : S)$	$\rightarrow$	$(M\,N, \mathbf{c}, S)$
(10)	$(B, \mathbf{c}, \mathsf{C}_2(x) : S)$	$\rightarrow$	$(\lambda x.B, \mathbf{c}, S)$
(11)	$(\lambda x.B, \mathbf{c}, \mathsf{C}_3(N) : S)$	$\rightarrow$	
	if $S \equiv \mathsf{C}_{(1\|3)}(\_) : S'$ then $([N/x]B, \mathbf{w}, S)$		
			else $([N/x]B, \mathbf{n}, S)$
(12)	$(M \not\equiv \lambda x.B, \mathbf{c}, \mathsf{C}_3(N) : S)$	$\rightarrow$	$(M, \mathbf{n}, \mathsf{C}_4(N) : S)$
(13)	$(M, \mathbf{c}, \mathsf{C}_4(N) : S)$	$\rightarrow$	$(N, \mathbf{n}, \mathsf{C}_5(M) : S)$
(14)	$(N, \mathbf{c}, \mathsf{C}_5(M) : S)$	$\rightarrow$	$(M\,N, \mathbf{c}, S)$
(15)	$(T, \mathbf{c}, \mathsf{C}_0)$	$\rightarrow$	T

(1)	T	$\to$	$(T, \mathbf{n}, C_0)$
(2)	$(x, \mathbf{n}, S)$	$\to$	$(x, \mathbf{c}, S)$
(3)	$(\lambda x.B, \mathbf{n}, S)$	$\to$	$(B, \mathbf{n}, C_2(x) : S)$
(4)	$(M\,N, \mathbf{n}, S)$	$\to$	$(M, \mathbf{w}, C_3(N) : S)$
(5)	$(x, \mathbf{w}, S)$	$\to$	$(x, \mathbf{c}, S)$
(6)	$(\lambda x.B, \mathbf{w}, S)$	$\to$	$(\lambda x.B, \mathbf{c}, S)$
(7)	$(M\,N, \mathbf{w}, S)$	$\to$	$(M, \mathbf{w}, C_1(N) : S)$
(8)	$(\lambda x.B, \mathbf{c}, C_1(N) : S)$	$\to$	$([N/x]B, \mathbf{w}, S)$
(9)	$(M \not\equiv \lambda x.B, \mathbf{c}, C_1(N) : S)$	$\to$	$(M\,N, \mathbf{c}, S)$
(10)	$(B, \mathbf{c}, C_2(x) : S)$	$\to$	$(\lambda x.B, \mathbf{c}, S)$
(11)	$(\lambda x.B, \mathbf{c}, C_3(N) : S)$	$\to$	$([N/x]B, \mathbf{n}, S)$
(12)	$(M \not\equiv \lambda x.B, \mathbf{c}, C_3(N) : S)$	$\to$	$(M, \mathbf{n}, C_4(N) : S)$
(13)	$(M, \mathbf{c}, C_4(N) : S)$	$\to$	$(N, \mathbf{n}, C_5(M) : S)$
(14)	$(N, \mathbf{c}, C_5(M) : S)$	$\to$	$(M\,N, \mathbf{c}, S)$
(15)	$(T, \mathbf{c}, C_0)$	$\to$	T

Abstract machine

(1)	T	$\to$	$(T, \mathbf{n}, C_0)$
(2)	$(x, \mathbf{n}, S)$	$\to$	$(x, \mathbf{c}, S)$
(3)	$(\lambda x.B, \mathbf{n}, S)$	$\to$	$(B, \mathbf{n}, C_2(x) : S)$
(4)	$(M\ N, \mathbf{n}, S)$	$\to$	$(M, \mathbf{w}, C_3(N) : S)$
(5)	$(x, \mathbf{w}, S)$	$\to$	$(x, \mathbf{c}, S)$
(6)	$(\lambda x.B, \mathbf{w}, S)$	$\to$	$(\lambda x.B, \mathbf{c}, S)$
(7)	$(M\ N, \mathbf{w}, S)$	$\to$	$(M, \mathbf{w}, C_1(N) : S)$
(8)	$(\lambda x.B, \mathbf{c}, C_1(N) : S)$	$\to$	$([N/x]B, \mathbf{w}, S)$
(9)	$(M \not\equiv \lambda x.B, \mathbf{c}, C_1(N) : S)$	$\to$	$(M\ N, \mathbf{c}, S)$
(10)	$(B, \mathbf{c}, C_2(x) : S)$	$\to$	$(\lambda x.B, \mathbf{c}, S)$
(11)	$(\lambda x.B, \mathbf{c}, C_3(N) : S)$	$\to$	$([N/x]B, \mathbf{n}, S)$
(12)	$(M \not\equiv \lambda x.B, \mathbf{c}, C_3(N) : S)$	$\to$	$(M, \mathbf{n}, C_4(N) : S)$
(13)	$(M, \mathbf{c}, C_4(N) : S)$	$\to$	$(N, \mathbf{n}, C_5(M) : S)$
(14)	$(N, \mathbf{c}, C_5(M) : S)$	$\to$	$(M\ N, \mathbf{c}, S)$
(15)	$(T, \mathbf{c}, C_0)$	$\to$	T

Abstract machine

(1)	T	$\to$	$(T, \mathbf{n}, C_0)$
(2)	$(x, \mathbf{n}, S)$	$\to$	$(x, \mathbf{c}, S)$
(3)	$(\lambda x.B, \mathbf{n}, S)$	$\to$	$(B, \mathbf{n}, C_2(x) : S)$
(4)	$(M\ N, \mathbf{n}, S)$	$\to$	$(M, \mathbf{w}, C_3(N) : S)$
(5)	$(x, \mathbf{w}, S)$	$\to$	$(x, \mathbf{c}, S)$
(6)	$(\lambda x.B, \mathbf{w}, S)$	$\to$	$(\lambda x.B, \mathbf{c}, S)$
(7)	$(M\ N, \mathbf{w}, S)$	$\to$	$(M, \mathbf{w}, C_1(N) : S)$
(8)	$(\lambda x.B, \mathbf{c}, C_1(N) : S)$	$\to$	$([N/x]B, \mathbf{w}, S)$
(9)	$(M \not\equiv \lambda x.B, \mathbf{c}, C_1(N) : S)$	$\to$	$(M\ N, \mathbf{c}, S)$
(10)	$(B, \mathbf{c}, C_2(x) : S)$	$\to$	$(\lambda x.B, \mathbf{c}, S)$
(11)	$(\lambda x.B, \mathbf{c}, C_3(N) : S)$	$\to$	$([N/x]B, \mathbf{n}, S)$
(12)	$(M \not\equiv \lambda x.B, \mathbf{c}, C_3(N) : S)$	$\to$	$(M, \mathbf{n}, C_4(N) : S)$
(13)	$(M, \mathbf{c}, C_4(N) : S)$	$\to$	$(N, \mathbf{n}, C_5(M) : S)$
(14)	$(N, \mathbf{c}, C_5(M) : S)$	$\to$	$(M\ N, \mathbf{c}, S)$
(15)	$(T, \mathbf{c}, C_0)$	$\to$	T

Reduction-free normaliser

$$\frac{}{x \Downarrow_{bn} x}\ (\textsc{Bn-Var}) \qquad \frac{}{\lambda x.B \Downarrow_{bn} \lambda x.B}\ (\textsc{Bn-Abs})$$

$$\frac{M \Downarrow_{bn} M' \quad M' \equiv \lambda x.B \quad [N/x]B \Downarrow_{bn} B'}{M\,N \Downarrow_{bn} B'}\ (\textsc{Bn-Con})$$

$$\frac{M \Downarrow_{bn} M' \quad M' \not\equiv \lambda x.B}{M\,N \Downarrow_{bn} M'\,N}\ (\textsc{Bn-Neu})$$

$$\frac{}{x \Downarrow_{no} x}\ (\textsc{No-Var}) \qquad \frac{B \Downarrow_{no} B'}{\lambda x.B \Downarrow_{no} \lambda x.B'}\ (\textsc{No-Abs})$$

$$\frac{M \Downarrow_{bn} M' \quad M' \equiv \lambda x.B \quad [N/x]B \Downarrow_{no} B'}{M\,N \Downarrow_{no} B'}\ (\textsc{No-Con})$$

$$\frac{M \Downarrow_{bn} M' \quad M' \not\equiv \lambda x.B \quad M' \Downarrow_{no} M'' \quad N \Downarrow_{no} N'}{M\,N \Downarrow_{no} M''\,N'}\ (\textsc{No-Neu})$$

Reduction-free normaliser

$$\frac{}{x \Downarrow_{bn} x}\ (\textsc{Bn-Var}) \qquad \frac{}{\lambda x.B \Downarrow_{bn} \lambda x.B}\ (\textsc{Bn-Abs})$$

$$\frac{M \Downarrow_{bn} M' \quad M' \equiv \lambda x.B \quad [N/x]B \Downarrow_{bn} B'}{M\ N \Downarrow_{bn} B'}\ (\textsc{Bn-Con})$$

$$\frac{M \Downarrow_{bn} M' \quad M' \not\equiv \lambda x.B}{M\ N \Downarrow_{bn} M'\ N}\ (\textsc{Bn-Neu})$$

$$\frac{}{x \Downarrow_{no} x}\ (\textsc{No-Var}) \qquad \frac{B \Downarrow_{no} B'}{\lambda x.B \Downarrow_{no} \lambda x.B'}\ (\textsc{No-Abs})$$

$$\frac{M \Downarrow_{bn} M' \quad M' \equiv \lambda x.B \quad [N/x]B \Downarrow_{no} B'}{M\ N \Downarrow_{no} B'}\ (\textsc{No-Con})$$

$$\frac{M \Downarrow_{bn} M' \quad M' \not\equiv \lambda x.B \quad M' \Downarrow_{no} M'' \quad N \Downarrow_{no} N'}{M\ N \Downarrow_{no} M''\ N'}\ (\textsc{No-Neu})$$

Code in ML (http://babel.ls.fi.upm.es/~agarcia/papers/PEPM13)

(≃ 800 LOC)

```
datatype term = IND of int | LAM of term | APP of term * term
datatype whnf = FUN of term | ACC of int * term list
datatype redex = SUB of term * term
datatype found = WHNF of whnf | RED of redex
datatype continuation = C0
                      | C1 of term * continuation
                      | C2 of continuation
                      | C3 of term * continuation
                      | C4 of term * continuation
                      | C5 of continuation * whnf
datatype whnf_or_decomposition = WHNF of whnf | DEC of redex * continuation

(*  decompose_cont : continuation * whnf -> whnf_or_decomposition *)
fun decompose_cont (C0, w)          = WHNF w
  | decompose_cont (C1 (n, k), wm)  = (case wm of (FUN b) => DEC (SUB (b, n), k)
                                                | _       => decompose_cont (k, apply_acc (wm, n)))
  | decompose_cont (C2 k, wb)       = decompose_cont (k, FUN (embed wb))
  | decompose_cont (C3 (n, k), wm)  = (case wm of (FUN b) => DEC (SUB (b, n), k)
                                                | _       => decompose_nf (embed wm, C4 (n, k)))
  | decompose_cont (C4 (n, k), nm)  = decompose_nf (n, C5 (k, nm))
  | decompose_cont (C5 (k, nm), nn) = decompose_cont (k, apply_acc (nm, embed nn))

(*  decompose_whnf : term * continuation -> whnf_or_decomposition *)
and decompose_whnf (IND n, k)      = decompose_cont (k, ACC (n, []))
  | decompose_whnf (LAM b, k)      = decompose_cont (k, FUN b)
  | decompose_whnf (APP (m, n), k) = decompose_whnf (m, C1 (n, k))

(*  decompose_nf : term * continuation -> whnf_or_decomposition *)
and decompose_nf (IND n, k)      = decompose_cont (k, ACC (n, []))
  | decompose_nf (LAM b, k)      = decompose_nf (b, C2 k)
  | decompose_nf (APP (m, n), k) = decompose_whnf (m, C3 (n, k))

(*  decompose : term -> whnf_or_decomposition *)
fun decompose t = decompose_nf (t, C0)
```

From normal order to full Krivine machine (KN, 2007) (21/22)

$$\text{Red-free normal} \overset{\text{clos.conv.}}{\Longrightarrow} \text{Hyb-norm} \overset{\text{exp.ctrl.}}{\Longrightarrow} \text{Gen-norm} \overset{\text{functional}}{\Longleftrightarrow} \text{Eval/apply machine} \overset{\text{inl.simp.}}{\Longrightarrow} \text{KN}$$

Thanks!