

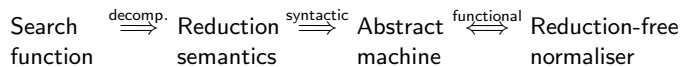
A Syntactic and Functional Correspondence
between
Reduction Semantics and Reduction-Free
Full Normalisers

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Pure untyped lambda calculus:

$$\Lambda ::= x \mid (\lambda x. \Lambda) \mid (\Lambda \Lambda)$$

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$$(\lambda x. B)N \rightarrow_{\beta} [N/x]B$$

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$$\begin{aligned} \text{WHNF} & ::= \lambda x. \Lambda \\ & \mid x \{\Lambda\}^* \end{aligned}$$

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Reduction:

$$\underline{(\lambda x. \lambda y. x)((\lambda x. x)z)} \rightarrow_{\beta} \lambda y. \underline{(\lambda x. x)z} \rightarrow_{\beta} \lambda y. z$$

$$\begin{aligned} \text{WHNF} & ::= \lambda x. \Lambda \\ & \mid x \{\Lambda\}^* \end{aligned}$$

$$\begin{aligned} \text{NF} & ::= \lambda x. \text{NF} \\ & \mid x \{\text{NF}\}^* \end{aligned}$$

Existing techniques

$$\frac{}{(\lambda x. B)N \rightarrow_{bn} [N/x]B} (\beta)$$

$$\frac{M \rightarrow_{bn} M'}{MN \rightarrow_{bn} M'N} (\mu)$$

$$\overline{(\lambda x. B)N \rightarrow_{bn} [N/x]B} \quad (\beta)$$

$$\frac{M \rightarrow_{bn} M'}{M N \rightarrow_{bn} M' N} \quad (\mu)$$

⇓ decomposition

$$\overline{(\lambda x. B)N \rightarrow_{bn} [N/x]B} \quad (\beta)$$

$$\frac{M \rightarrow_{bn} M'}{M N \rightarrow_{bn} M' N} \quad (\mu)$$

⇓ (CPS, defunc, ...)

$$\frac{}{(\lambda x. B)N \rightarrow_{bn} [N/x]B} (\beta)$$

$$\frac{M \rightarrow_{bn} M'}{MN \rightarrow_{bn} M'N} (\mu)$$

↓ (CPS, defunc, ...)

Contexts: $C_{bn}[] ::= [] \mid C_{bn}[] \wedge$

β -rule: $C_{bn}[(\lambda x. B)N] \rightarrow_{\beta} C_{bn}[[N/x]B]$

$$\overline{(\lambda x. B)N \rightarrow_{bn} [N/x]B} \quad (\beta)$$

$$\frac{M \rightarrow_{bn} M'}{MN \rightarrow_{bn} M'N} \quad (\mu)$$

↓ (CPS, defunc, ...) **redex at the root**

Contexts: $C_{bn}[\] ::= [\] \mid C_{bn}[\] \wedge$
 β -rule: $C_{bn}[(\lambda x. B)N] \rightarrow_{\beta} C_{bn}[[N/x]B]$

$$\overline{(\lambda x. B)N \rightarrow_{bn} [N/x]B} \quad (\beta)$$

$$\frac{M \rightarrow_{bn} M'}{MN \rightarrow_{bn} M'N} \quad (\mu)$$

↓ (CPS, defunc, ...) **recurring over the operator**

Contexts: $C_{bn}[] ::= [] \mid C_{bn}[] \wedge$
 β -rule: $C_{bn}[(\lambda x. B)N] \rightarrow_{\beta} C_{bn}[[N/x]B]$

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$$\frac{M \rightarrow_{bn} M'}{MN \rightarrow_{bn} M'N} \quad (\mu)$$

↓ (CPS, defunc, ...)

Contexts: $C_{bn}[] ::= [] \mid C_{bn}[] \wedge$
 β -rule: $C_{bn}[(\lambda x. B)N] \rightarrow_{\beta} C_{bn}[[N/x]B]$

↓ syntactic

$$\overline{(\lambda x. B)N \rightarrow_{bn} [N/x]B} \quad (\beta)$$

$$\frac{M \rightarrow_{bn} M'}{MN \rightarrow_{bn} M'N} \quad (\mu)$$

↓ (CPS, defunc, ...)

Contexts: $C_{bn}[] ::= [] \mid C_{bn}[] \wedge$
 β -rule: $C_{bn}[(\lambda x. B)N] \rightarrow_{\beta} C_{bn}[[N/x]B]$

↓ (refocus, LWF, inline)

$$\overline{(\lambda x. B) N \rightarrow_{bn} [N/x] B} \quad (\beta)$$

$$\frac{M \rightarrow_{bn} M'}{M N \rightarrow_{bn} M' N} \quad (\mu)$$

↓ (CPS, defunc, ...)

Contexts: $C_{bn}[] ::= [] \mid C_{bn}[] \wedge$
 β -rule: $C_{bn}[(\lambda x. B) N] \rightarrow_{\beta} C_{bn}[[N/x] B]$

↓ (refocus, LWF, inline)

(1)	T	\rightarrow	(T, \mathbf{e}, C_0)
(2)	(x, \mathbf{e}, S)	\rightarrow	(x, \mathbf{c}, S)
(3)	$(\lambda x. B, \mathbf{e}, S)$	\rightarrow	$(\lambda x. B, \mathbf{c}, S)$
(4)	$(M N, \mathbf{e}, S)$	\rightarrow	$(M, \mathbf{e}, C_1(N) : S)$
(5)	$(\lambda x. B, \mathbf{c}, C_1(N) : S)$	\rightarrow	$([N/x] B, \mathbf{e}, S)$
(6)	$(M \neq \lambda x. B, \mathbf{c}, C_1(N) : S)$	\rightarrow	$(M N, \mathbf{c}, S)$
(7)	(T, \mathbf{c}, C_0)	\rightarrow	T

$$\overline{(\lambda x.B)N \rightarrow_{bn} [N/x]B} \quad (\beta)$$

$$\frac{M \rightarrow_{bn} M'}{MN \rightarrow_{bn} M'N} \quad (\mu)$$

↓ (CPS, defunc, ...)

Contexts: $C_{bn}[] ::= [] \mid C_{bn}[] \wedge$
 β -rule: $C_{bn}[(\lambda x.B)N] \rightarrow_{\beta} C_{bn}[[N/x]B]$

↓ (refocus, LWF, inline) init

(1)	$T \rightarrow (T, e, C_0)$
(2)	$(x, e, S) \rightarrow (x, c, S)$
(3)	$(\lambda x.B, e, S) \rightarrow (\lambda x.B, c, S)$
(4)	$(MN, e, S) \rightarrow (M, e, C_1(N) : S)$
(5)	$(\lambda x.B, c, C_1(N) : S) \rightarrow ([N/x]B, e, S)$
(6)	$(M \neq \lambda x.B, c, C_1(N) : S) \rightarrow (MN, c, S)$
(7)	$(T, c, C_0) \rightarrow T$

$$\overline{(\lambda x. B)N \rightarrow_{bn} [N/x]B} \quad (\beta)$$

$$\frac{M \rightarrow_{bn} M'}{MN \rightarrow_{bn} M'N} \quad (\mu)$$

↓ (CPS, defunc, ...)

Contexts: $C_{bn}[] ::= [] \mid C_{bn}[] \wedge$
 β -rule: $C_{bn}[(\lambda x. B)N] \rightarrow_{\beta} C_{bn}[[N/x]B]$

↓ (refocus, LWF, inline) empty

(1)	T	\rightarrow	(T, \mathbf{e}, C_0)
(2)	(x, \mathbf{e}, S)	\rightarrow	(x, \mathbf{c}, S)
(3)	$(\lambda x. B, \mathbf{e}, S)$	\rightarrow	$(\lambda x. B, \mathbf{c}, S)$
(4)	(MN, \mathbf{e}, S)	\rightarrow	$(M, \mathbf{e}, C_1(N) : S)$
(5)	$(\lambda x. B, \mathbf{c}, C_1(N) : S)$	\rightarrow	$([N/x]B, \mathbf{e}, S)$
(6)	$(M \neq \lambda x. B, \mathbf{c}, C_1(N) : S)$	\rightarrow	(MN, \mathbf{c}, S)
(7)	(T, \mathbf{c}, C_0)	\rightarrow	T

$$\overline{(\lambda x. B)N \rightarrow_{bn} [N/x]B} \quad (\beta)$$

$$\frac{M \rightarrow_{bn} M'}{MN \rightarrow_{bn} M'N} \quad (\mu)$$

↓ (CPS, defunc, ...)

Contexts: $C_{bn}[] ::= [] \mid C_{bn}[] \wedge$
 β -rule: $C_{bn}[(\lambda x. B)N] \rightarrow_{\beta} C_{bn}[[N/x]B]$

↓ (refocus, LWF, inline) operator

(1)	T	\rightarrow	(T, \mathbf{e}, C_0)
(2)	(x, \mathbf{e}, S)	\rightarrow	(x, \mathbf{c}, S)
(3)	$(\lambda x. B, \mathbf{e}, S)$	\rightarrow	$(\lambda x. B, \mathbf{c}, S)$
(4)	(MN, \mathbf{e}, S)	\rightarrow	$(M, \mathbf{e}, C_1(N) : S)$
(5)	$(\lambda x. B, \mathbf{c}, C_1(N) : S)$	\rightarrow	$([N/x]B, \mathbf{e}, S)$
(6)	$(M \neq \lambda x. B, \mathbf{c}, C_1(N) : S)$	\rightarrow	(MN, \mathbf{c}, S)
(7)	(T, \mathbf{c}, C_0)	\rightarrow	T

$$\overline{(\lambda x. B) N \rightarrow_{bn} [N/x] B} \quad (\beta)$$

$$\frac{M \rightarrow_{bn} M'}{M N \rightarrow_{bn} M' N} \quad (\mu)$$

↓ (CPS, defunc, ...)

Contexts: $C_{bn}[] ::= [] \mid C_{bn}[] \wedge$
 β -rule: $C_{bn}[(\lambda x. B) N] \rightarrow_{\beta} C_{bn}[[N/x] B]$

↓ (refocus, LWF, inline) eval

(1)	T	\rightarrow	(T, \mathbf{e}, C_0)
(2)	(x, \mathbf{e}, S)	\rightarrow	(x, \mathbf{c}, S)
(3)	$(\lambda x. B, \mathbf{e}, S)$	\rightarrow	$(\lambda x. B, \mathbf{c}, S)$
(4)	$(M N, \mathbf{e}, S)$	\rightarrow	$(M, \mathbf{e}, C_1(N) : S)$
(5)	$(\lambda x. B, \mathbf{c}, C_1(N) : S)$	\rightarrow	$([N/x] B, \mathbf{e}, S)$
(6)	$(M \neq \lambda x. B, \mathbf{c}, C_1(N) : S)$	\rightarrow	$(M N, \mathbf{c}, S)$
(7)	(T, \mathbf{c}, C_0)	\rightarrow	T

$$\overline{(\lambda x. B) N \rightarrow_{bn} [N/x] B} \quad (\beta)$$

$$\frac{M \rightarrow_{bn} M'}{M N \rightarrow_{bn} M' N} \quad (\mu)$$

↓ (CPS, defunc, ...)

Contexts: $C_{bn}[] ::= [] \mid C_{bn}[] \Lambda$
 β -rule: $C_{bn}[(\lambda x. B) N] \rightarrow_{\beta} C_{bn}[[N/x] B]$

↓ (refocus, LWF, inline)

apply cont

(1)	T	\rightarrow	(T, \mathbf{e}, C_0)
(2)	(x, \mathbf{e}, S)	\rightarrow	(x, \mathbf{c}, S)
(3)	$(\lambda x. B, \mathbf{e}, S)$	\rightarrow	$(\lambda x. B, \mathbf{c}, S)$
(4)	$(M N, \mathbf{e}, S)$	\rightarrow	$(M, \mathbf{e}, C_1(N) : S)$
(5)	$(\lambda x. B, \mathbf{c}, C_1(N) : S)$	\rightarrow	$([N/x] B, \mathbf{e}, S)$
(6)	$(M \neq \lambda x. B, \mathbf{c}, C_1(N) : S)$	\rightarrow	$(M N, \mathbf{c}, S)$
(7)	(T, \mathbf{c}, C_0)	\rightarrow	T

$$\overline{(\lambda x. B)N \rightarrow_{bn} [N/x]B} \quad (\beta)$$

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Contexts: $C_{bn}[] ::= [] \mid C_{bn}[] \wedge$
 β -rule: $C_{bn}[(\lambda x. B)N] \rightarrow_{\beta} C_{bn}[[N/x]B]$

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(1)	T	\rightarrow	(T, \mathbf{e}, C_0)
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(5)	$(\lambda x. B, \mathbf{c}, C_1(N) : S)$	\rightarrow	$([N/x]B, \mathbf{e}, S)$
(6)	$(M \neq \lambda x. B, \mathbf{c}, C_1(N) : S)$	\rightarrow	(MN, \mathbf{c}, S)
(7)	(T, \mathbf{c}, C_0)	\rightarrow	T

↕ functional

$$\overline{(\lambda x.B)N \rightarrow_{bn} [N/x]B} \quad (\beta)$$

$$\frac{M \rightarrow_{bn} M'}{MN \rightarrow_{bn} M'N} \quad (\mu)$$

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 β -rule: $C_{bn}[(\lambda x.B)N] \rightarrow_{\beta} C_{bn}[[N/x]B]$

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(4)	$(MN, \mathbf{e}, S) \rightarrow (M, \mathbf{e}, C_1(N) : S)$
(5)	$(\lambda x.B, \mathbf{c}, C_1(N) : S) \rightarrow ([N/x]B, \mathbf{e}, S)$
(6)	$(M \neq \lambda x.B, \mathbf{c}, C_1(N) : S) \rightarrow (MN, \mathbf{c}, S)$
(7)	$(T, \mathbf{c}, C_0) \rightarrow T$

↕ (refunc, CPS⁻¹)

$$\overline{(\lambda x. B)N \rightarrow_{bn} [N/x]B} \quad (\beta)$$

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(7)	$(T, c, C_0) \rightarrow T$

↑ (refunc, CPS⁻¹)

$$\overline{x \downarrow_{bn} x} \quad (\text{BN-VAR})$$

$$\overline{\lambda x. B \downarrow_{bn} \lambda x. B} \quad (\text{BN-ABS})$$

$$\frac{M \downarrow_{bn} M' \quad M' \equiv \lambda x. B \quad [N/x]B \downarrow_{bn} B'}{MN \downarrow_{bn} B'} \quad (\text{BN-CON})$$

$$\frac{M \downarrow_{bn} M' \quad M' \neq \lambda x. B}{MN \downarrow_{bn} M'N} \quad (\text{BN-NEU})$$

$$\overline{(\lambda x. B) N \rightarrow_{bn} [N/x] B} \quad (\beta)$$

$$\frac{M \rightarrow_{bn} M'}{M N \rightarrow_{bn} M' N} \quad (\mu)$$

↓ (CPS, defunc, ...)

Contexts: $C_{bn}[] ::= [] \mid C_{bn}[] \wedge$
 β -rule: $C_{bn}[(\lambda x. B) N] \rightarrow_{\beta} C_{bn}[[N/x] B]$

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(1)	$T \rightarrow (T, e, C_0)$
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(4)	$(M N, e, S) \rightarrow (M, e, C_1(N) : S)$
(5)	$(\lambda x. B, c, C_1(N) : S) \rightarrow ([N/x] B, e, S)$
(6)	$(M \neq \lambda x. B, c, C_1(N) : S) \rightarrow (M N, c, S)$
(7)	$(T, c, C_0) \rightarrow T$

↑ (refunc, CPS⁻¹)

irred

$$\overline{x \downarrow_{bn} x} \quad (\text{BN-VAR})$$

$$\overline{\lambda x. B \downarrow_{bn} \lambda x. B} \quad (\text{BN-ABS})$$

$$\frac{M \downarrow_{bn} M' \quad M' \equiv \lambda x. B \quad [N/x] B \downarrow_{bn} B'}{M N \downarrow_{bn} B'} \quad (\text{BN-CON})$$

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$$\overline{(\lambda x. B) N \rightarrow_{bn} [N/x] B} \quad (\beta)$$

$$\frac{M \rightarrow_{bn} M'}{M N \rightarrow_{bn} M' N} \quad (\mu)$$

↓ (CPS, defunc, ...)

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(7)	$(T, c, C_0) \rightarrow T$

↑ (refunc, CPS⁻¹)

contract

$$\overline{x \downarrow_{bn} x} \quad (\text{BN-VAR})$$

$$\overline{\lambda x. B \downarrow_{bn} \lambda x. B} \quad (\text{BN-ABS})$$

$$\frac{M \downarrow_{bn} M' \quad M' \equiv \lambda x. B \quad [N/x] B \downarrow_{bn} B'}{M N \downarrow_{bn} B'} \quad (\text{BN-CON})$$

$$\frac{M \downarrow_{bn} M' \quad M' \neq \lambda x. B}{M N \downarrow_{bn} M' N} \quad (\text{BN-NEU})$$

$$\frac{}{(\lambda x.B)N \rightarrow_{bn} [N/x]B} (\beta) \qquad \frac{M \rightarrow_{bn} M'}{MN \rightarrow_{bn} M'N} (\mu)$$

↓ (CPS, defunc, ...)

Contexts: $C_{bn}[] ::= [] \mid C_{bn}[] \wedge$
 β -rule: $C_{bn}[(\lambda x.B)N] \rightarrow_{\beta} C_{bn}[[N/x]B]$

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(5)	$(\lambda x.B, c, C_1(N) : S) \rightarrow ([N/x]B, e, S)$
(6)	$(M \neq \lambda x.B, c, C_1(N) : S) \rightarrow (MN, c, S)$
(7)	$(T, c, C_0) \rightarrow T$

↑ (refunc, CPS⁻¹)

neutral

$$\frac{}{x \Downarrow_{bn} x} (\text{BN-VAR})$$

$$\frac{}{\lambda x.B \Downarrow_{bn} \lambda x.B} (\text{BN-ABS})$$

$$\frac{M \Downarrow_{bn} M' \quad M' \equiv \lambda x.B \quad [N/x]B \Downarrow_{bn} B'}{MN \Downarrow_{bn} B'} (\text{BN-CON})$$

$$\frac{M \Downarrow_{bn} M' \quad M' \neq \lambda x.B}{MN \Downarrow_{bn} M'N} (\text{BN-NEU})$$

$$\overline{(\lambda x.B)N \rightarrow_{bn} [N/x]B} \quad (\beta)$$

$$\frac{M \rightarrow_{bn} M'}{MN \rightarrow_{bn} M'N} \quad (\mu)$$

↓ (CPS, defunc, ...)

Contexts: $C_{bn}[] ::= [] \mid C_{bn}[] \wedge$
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(3)	$(\lambda x.B, e, S) \rightarrow (\lambda x.B, c, S)$
(4)	$(MN, e, S) \rightarrow (M, e, C_1(N) : S)$
(5)	$(\lambda x.B, c, C_1(N) : S) \rightarrow ([N/x]B, e, S)$
(6)	$(M \neq \lambda x.B, c, C_1(N) : S) \rightarrow (MN, c, S)$
(7)	$(T, c, C_0) \rightarrow T$

↑ (refunc, CPS⁻¹)

$$\overline{x \downarrow_{bn} x} \quad (\text{BN-VAR})$$

$$\overline{\lambda x.B \downarrow_{bn} \lambda x.B} \quad (\text{BN-ABS})$$

$$\frac{M \downarrow_{bn} M' \quad M' \equiv \lambda x.B \quad [N/x]B \downarrow_{bn} B'}{MN \downarrow_{bn} B'} \quad (\text{BN-CON})$$

$$\frac{M \downarrow_{bn} M' \quad M' \neq \lambda x.B}{MN \downarrow_{bn} M'N} \quad (\text{BN-NEU})$$

However...

Normal order is hybrid

(7/22)

$$\frac{}{x \Downarrow_{bn} x} \text{ (BN-VAR)} \qquad \frac{}{\lambda x.B \Downarrow_{bn} \lambda x.B} \text{ (BN-ABS)}$$

$$\frac{M \Downarrow_{bn} M' \quad M' \equiv \lambda x.B \quad [N/x]B \Downarrow_{bn} B'}{M N \Downarrow_{bn} B'} \text{ (BN-CON)}$$

$$\frac{M \Downarrow_{bn} M' \quad M' \not\equiv \lambda x.B}{M N \Downarrow_{bn} M' N} \text{ (BN-NEU)}$$

$$\frac{}{x \Downarrow_{no} x} \text{ (NO-VAR)} \qquad \frac{B \Downarrow_{no} B'}{\lambda x.B \Downarrow_{no} \lambda x.B'} \text{ (NO-ABS)}$$

$$\frac{M \Downarrow_{bn} M' \quad M' \equiv \lambda x.B \quad [N/x]B \Downarrow_{no} B'}{M N \Downarrow_{no} B'} \text{ (NO-CON)}$$

$$\frac{M \Downarrow_{bn} M' \quad M' \not\equiv \lambda x.B \quad M' \Downarrow_{no} M'' \quad N \Downarrow_{no} N'}{M N \Downarrow_{no} M'' N'} \text{ (NO-NEU)}$$

$$\frac{}{x \Downarrow_{bn} x} \text{ (BN-VAR)}$$

$$\frac{}{\lambda x.B \Downarrow_{bn} \lambda x.B} \text{ (BN-ABS)}$$

$$\frac{M \Downarrow_{bn} M' \quad M' \equiv \lambda x.B \quad [N/x]B \Downarrow_{bn} B'}{M N \Downarrow_{bn} B'} \text{ (BN-CON)}$$

$$\frac{M \Downarrow_{bn} M' \quad M' \not\equiv \lambda x.B}{M N \Downarrow_{bn} M' N} \text{ (BN-NEU)}$$

$$\frac{}{x \Downarrow_{no} x} \text{ (NO-VAR)}$$

$$\frac{B \Downarrow_{no} B'}{\lambda x.B \Downarrow_{no} \lambda x.B'} \text{ (NO-ABS)}$$

$$\frac{M \Downarrow_{bn} M' \quad M' \equiv \lambda x.B \quad [N/x]B \Downarrow_{no} B'}{M N \Downarrow_{no} B'} \text{ (NO-CON)}$$

$$\frac{M \Downarrow_{bn} M' \quad M' \not\equiv \lambda x.B \quad M' \Downarrow_{no} M'' \quad N \Downarrow_{no} N'}{M N \Downarrow_{no} M'' N'} \text{ (NO-NEU)}$$

Our contribution

$$\frac{}{(\lambda x.B)N \rightarrow_{no} [N/x]B} (\beta) \quad \frac{M \notin \text{WHNF} \quad M \rightarrow_{no} M'}{MN \rightarrow_{no} M'N} (\mu 1)$$

$$\frac{M \in \text{WHNF} \quad M \not\equiv \lambda x.B \quad M \rightarrow_{no} M'}{MN \rightarrow_{no} M'N} (\mu 2)$$

$$\frac{M \in \text{NF} \quad M \not\equiv \lambda x.B \quad N \rightarrow_{no} N'}{MN \rightarrow_{no} MN'} (\nu)$$

$$\frac{B \rightarrow_{no} B'}{\lambda x.B \rightarrow_{no} \lambda x.B'} (\xi)$$

$$\frac{}{(\lambda x.B)N \rightarrow_{no} [N/x]B} (\beta) \quad \frac{M \notin \text{WHNF} \quad M \rightarrow_{no} M'}{MN \rightarrow_{no} M'N} (\mu 1)$$

$$\frac{M \in \text{WHNF} \quad M \not\equiv \lambda x.B \quad M \rightarrow_{no} M'}{MN \rightarrow_{no} M'N} (\mu 2)$$

$$\frac{M \in \text{NF} \quad M \not\equiv \lambda x.B \quad N \rightarrow_{no} N'}{MN \rightarrow_{no} MN'} (\nu)$$

$$\frac{B \rightarrow_{no} B'}{\lambda x.B \rightarrow_{no} \lambda x.B'} (\xi)$$

Neutrals in nf: $\text{NNF} ::= x \mid \text{NNF NF}$

Contexts: $C_{no}[] ::= [] \mid C_{bn}[] \wedge \mid \lambda x. C_{no}[] \mid C_{ne}[]$

$C_{bn}[] ::= [] \mid C_{bn}[] \wedge$

$C_{ne}[] ::= \text{NNF } C_{no}[] \mid C_{ne}[] \wedge$

β -rule: $C_{no}[(\lambda x. B)N] \rightarrow_{\beta} C_{no}[[N/x]B]$

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β -rule: $C_{no}[(\lambda x. B) N] \rightarrow_{\beta} C_{no}[[N/x]B]$

$$\begin{aligned}C_{no}[\] &::= [\] \mid C_{bn}[\] \wedge \mid \lambda x. C_{no}[\] \mid C_{ne}[\] \\C_{bn}[\] &::= [\] \mid C_{bn}[\] \wedge \\C_{ne}[\] &::= \text{NNF } C_{no}[\] \mid C_{ne}[\] \wedge\end{aligned}$$

$$C_{no}[] ::= [] \mid C_{bn}[] \wedge \mid \lambda x. C_{no}[] \mid C_{ne}[]$$

$$C_{bn}[] ::= [] \mid C_{bn}[] \wedge$$

$$C_{ne}[] ::= \text{NNF } C_{no}[] \mid C_{ne}[] \wedge$$

$$C_{no}[] \equiv C_0$$

$$C_{no}[] \Rightarrow \lambda x. C_{no}[] \equiv C_2(x)$$

$$C_{no}[] \Rightarrow C_{bn}[] \wedge \equiv C_3(\wedge)$$

$$C_{bn}[] \Rightarrow C_{bn}[] \wedge \equiv C_1(\wedge)$$

$$C_{bn}[] \Rightarrow [] \equiv \text{empty}$$

$$C_{no}[] \Rightarrow C_{ne}[] \equiv \text{empty}$$

$$C_{ne}[] \Rightarrow C_{ne}[] \wedge \equiv C_4(\wedge)$$

$$C_{ne}[] \Rightarrow \text{NNF } C_{no}[] \equiv C_5(\text{NNF})$$

$$C_{no}[] \Rightarrow [] \equiv \text{empty}$$

$$\begin{aligned}
 C_{no}[] & ::= \{\lambda x.\}^* [] \{\Lambda\}^* \Lambda \\
 & \quad | \{\lambda x.\}^* [] \\
 & \quad | \{\lambda x.\}^* \text{NNF } C_{no}[] \{\Lambda\}^*
 \end{aligned}$$

$$\begin{aligned}
 C_{no}[] & \equiv C_0 \\
 C_{no}[] \Rightarrow \lambda x. C_{no}[] & \equiv C_2(x) \\
 C_{no}[] \Rightarrow C_{bn}[] \Lambda & \equiv C_3(\Lambda) \\
 C_{bn}[] \Rightarrow C_{bn}[] \Lambda & \equiv C_1(\Lambda) \\
 C_{bn}[] \Rightarrow [] & \equiv \text{empty} \\
 C_{no}[] \Rightarrow C_{ne}[] & \equiv \text{empty} \\
 C_{ne}[] \Rightarrow C_{ne}[] \Lambda & \equiv C_4(\Lambda) \\
 C_{ne}[] \Rightarrow \text{NNF } C_{no}[] & \equiv C_5(\text{NNF}) \\
 C_{no}[] \Rightarrow [] & \equiv \text{empty}
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$$\begin{aligned}
 C_{no}[] & ::= \{\lambda x.\}^* [] \{\Lambda\}^* \wedge \\
 & \quad | \{\lambda x.\}^* [] \\
 & \quad | \{\lambda x.\}^* \text{NNF } C_{no}[] \{\Lambda\}^*
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 C_{no}[] & \equiv C_0 \\
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 C_{bn}[] \Rightarrow [] & \equiv \text{empty} \\
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 C_{no}[] \Rightarrow C_{ne}[] & \equiv \text{empty} \\
 C_{ne}[] \Rightarrow C_{ne}[] \wedge & \equiv C_4(\Lambda) \\
 C_{ne}[] \Rightarrow \text{NNF } C_{no}[] & \equiv \text{C}_5(\text{NNF}) \\
 C_{no}[] \Rightarrow [] & \equiv \text{empty}
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 C_{no}[] & ::= \{ \lambda x. \}^* [] \{ \Lambda \}^* \Lambda \\
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$$\begin{aligned} S &::= \{D\}^? C_0 \\ D &::= \{\{C_1(\Lambda) : \}^* C_3(\Lambda) : \}^? \{C_2(x) : \}^* \\ &\quad | \{D\}^? C_5(\text{NNF}) : \{C_4(\Lambda) : \}^* \{C_2(x) : \}^* \end{aligned}$$

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 S &::= \{D\}^? C_0 \\
 D &::= \{\{C_1(\Lambda) : \}^* C_3(\Lambda) : \}^? \{C_2(x) : \}^* \\
 &\quad | \{D\}^? C_5(\text{NNF}) : \{C_4(\Lambda) : \}^* \{C_2(x) : \}^*
 \end{aligned}$$

$$\begin{aligned}
 [] &\equiv C_0 \\
 [] N &\equiv C_3(N) : C_0 \\
 \lambda x. [] &\equiv C_2(x) : C_0 \\
 [] N'' N' N &\equiv C_1(N'') : C_1(N') : C_3(N) : C_0 \\
 \lambda x. \lambda y. [] N &\equiv C_3(N) : C_2(y) : C_2(x) : C_0 \\
 xM [] N &\equiv C_5(xM) : C_4(N) : C_0 \quad (xM \in \text{NNF}) \\
 \dots &
 \end{aligned}$$

Example with context stacks

(13/22)

$[(\lambda x.y(I I))I I, C_0]$

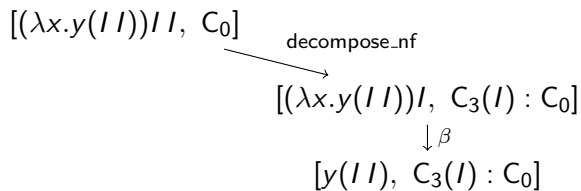
(remember $I \equiv \lambda x.x$)

Example with context stacks

(13/22)

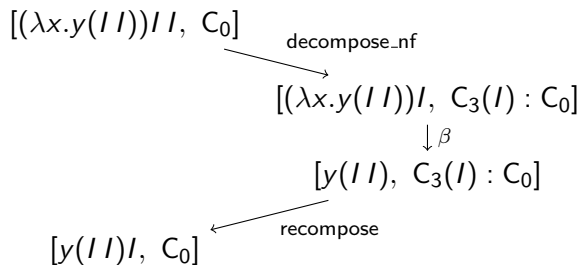
$[(\lambda x.y(I I))I I, C_0]$ $\xrightarrow{\text{decompose_nf}}$ $[(\lambda x.y(I I))I, C_3(I) : C_0]$

(remember $I \equiv \lambda x.x$)

(remember $I \equiv \lambda x.x$)

Example with context stacks

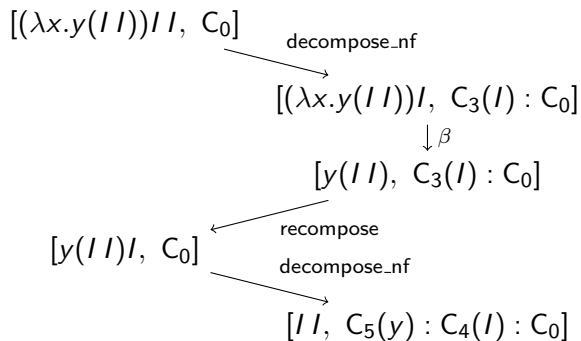
(13/22)



(remember $I \equiv \lambda x.x$)

Example with context stacks

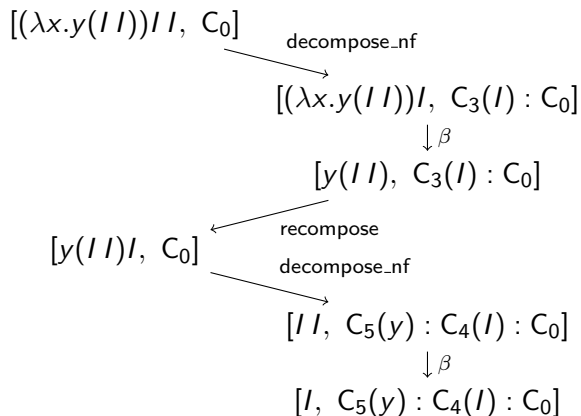
(13/22)



(remember $I \equiv \lambda x.x$)

Example with context stacks

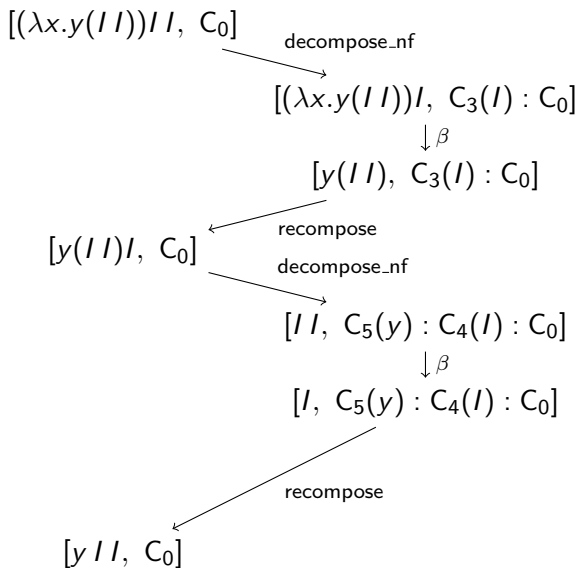
(13/22)



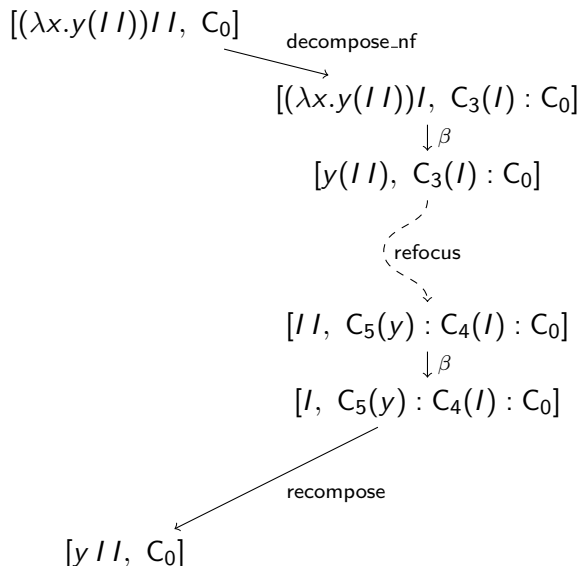
(remember $I \equiv \lambda x.x$)

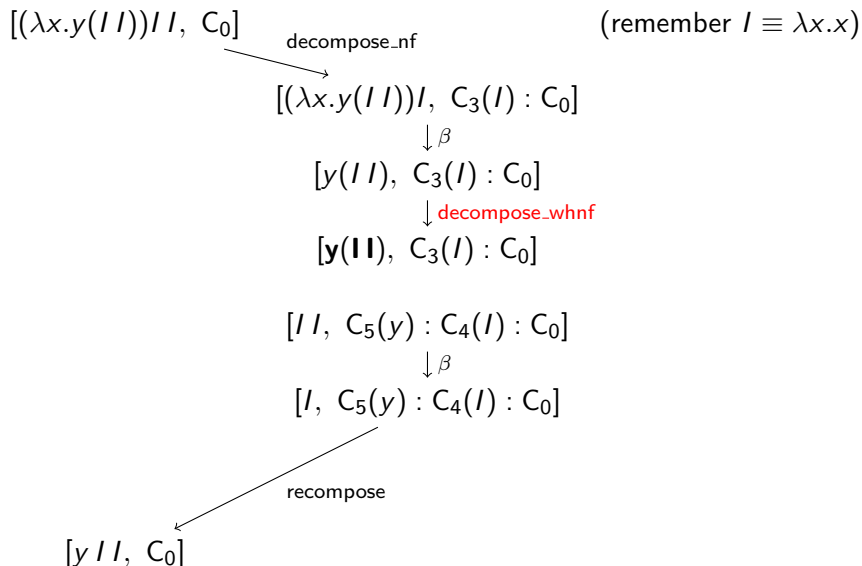
Example with context stacks

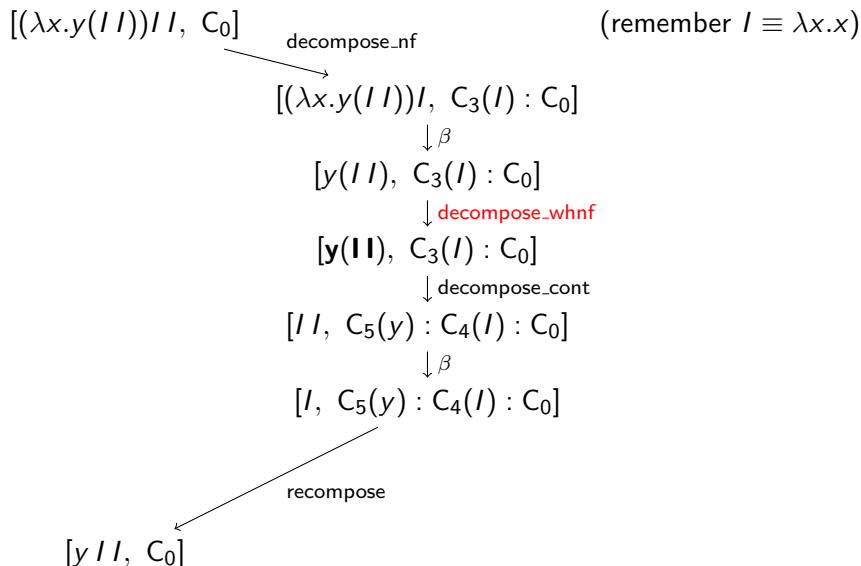
(13/22)

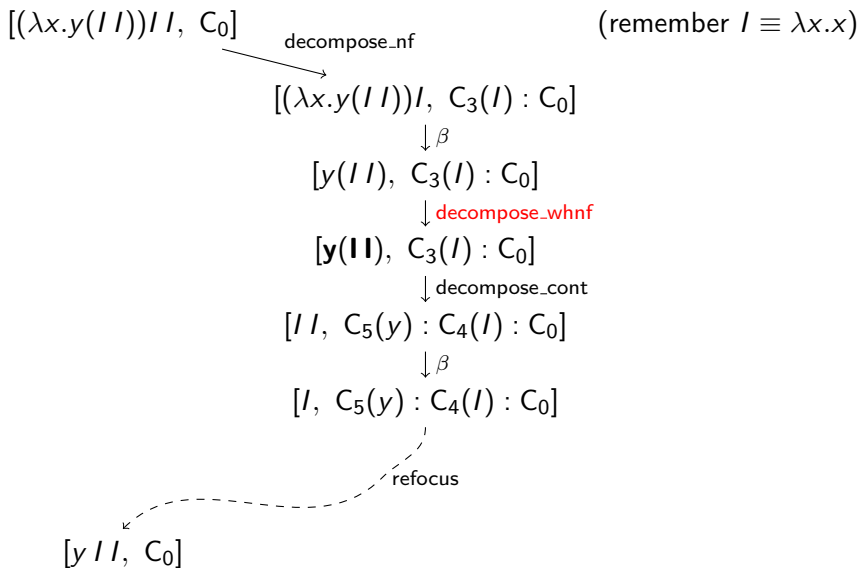


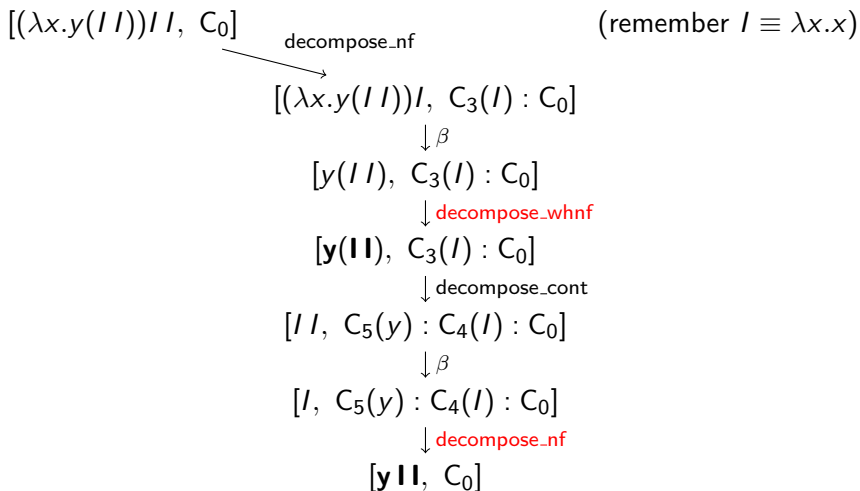
(remember $I \equiv \lambda x. x$)

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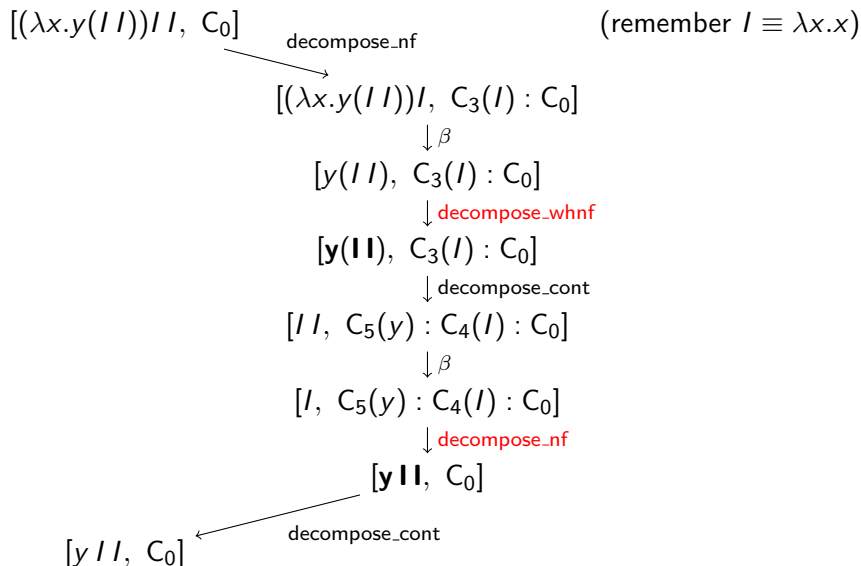








$[y I I, C_0]$



(1)	$T \rightarrow (T, \mathbf{n}, C_0)$
(2)	$(x, \mathbf{n}, S) \rightarrow (x, \mathbf{c}, S)$
(3)	$(\lambda x. B, \mathbf{n}, S) \rightarrow (B, \mathbf{n}, C_2(x) : S)$
(4)	$(M N, \mathbf{n}, S) \rightarrow (M, \mathbf{w}, C_3(N) : S)$
(5)	$(x, \mathbf{w}, S) \rightarrow (x, \mathbf{c}, S)$
(6)	$(\lambda x. B, \mathbf{w}, S) \rightarrow (\lambda x. B, \mathbf{c}, S)$
(7)	$(M N, \mathbf{w}, S) \rightarrow (M, \mathbf{w}, C_1(N) : S)$
(8)	$(\lambda x. B, \mathbf{c}, C_1(N) : S) \rightarrow$ if $S \equiv C_{(1 3)}(-) : S'$ then $([N/x]B, \mathbf{w}, S)$ else $([N/x]B, \mathbf{n}, S)$
(9)	$(M \neq \lambda x. B, \mathbf{c}, C_1(N) : S) \rightarrow (M N, \mathbf{c}, S)$
(10)	$(B, \mathbf{c}, C_2(x) : S) \rightarrow (\lambda x. B, \mathbf{c}, S)$
(11)	$(\lambda x. B, \mathbf{c}, C_3(N) : S) \rightarrow$ if $S \equiv C_{(1 3)}(-) : S'$ then $([N/x]B, \mathbf{w}, S)$ else $([N/x]B, \mathbf{n}, S)$
(12)	$(M \neq \lambda x. B, \mathbf{c}, C_3(N) : S) \rightarrow (M, \mathbf{n}, C_4(N) : S)$
(13)	$(M, \mathbf{c}, C_4(N) : S) \rightarrow (N, \mathbf{n}, C_5(M) : S)$
(14)	$(N, \mathbf{c}, C_5(M) : S) \rightarrow (M N, \mathbf{c}, S)$
(15)	$(T, \mathbf{c}, C_0) \rightarrow T$

(1)	T	\rightarrow	(T, \mathbf{n}, C_0)
(2)	(x, \mathbf{n}, S)	\rightarrow	(x, \mathbf{c}, S)
(3)	$(\lambda x. B, \mathbf{n}, S)$	\rightarrow	$(B, \mathbf{n}, C_2(x) : S)$
(4)	$(M N, \mathbf{n}, S)$	\rightarrow	$(M, \mathbf{w}, C_3(N) : S)$
(5)	(x, \mathbf{w}, S)	\rightarrow	(x, \mathbf{c}, S)
(6)	$(\lambda x. B, \mathbf{w}, S)$	\rightarrow	$(\lambda x. B, \mathbf{c}, S)$
(7)	$(M N, \mathbf{w}, S)$	\rightarrow	$(M, \mathbf{w}, C_1(N) : S)$
(8)	$(\lambda x. B, \mathbf{c}, C_1(N) : S)$	\rightarrow	<p style="color: red;">if $S \equiv C_{(1 3)}(-) : S'$ then $([N/x]B, \mathbf{w}, S)$ else $([N/x]B, \mathbf{n}, S)$</p>
(9)	$(M \neq \lambda x. B, \mathbf{c}, C_1(N) : S)$	\rightarrow	$(M N, \mathbf{c}, S)$
(10)	$(B, \mathbf{c}, C_2(x) : S)$	\rightarrow	$(\lambda x. B, \mathbf{c}, S)$
(11)	$(\lambda x. B, \mathbf{c}, C_3(N) : S)$	\rightarrow	<p style="color: red;">if $S \equiv C_{(1 3)}(-) : S'$ then $([N/x]B, \mathbf{w}, S)$ else $([N/x]B, \mathbf{n}, S)$</p>
(12)	$(M \neq \lambda x. B, \mathbf{c}, C_3(N) : S)$	\rightarrow	$(M, \mathbf{n}, C_4(N) : S)$
(13)	$(M, \mathbf{c}, C_4(N) : S)$	\rightarrow	$(N, \mathbf{n}, C_5(M) : S)$
(14)	$(N, \mathbf{c}, C_5(M) : S)$	\rightarrow	$(M N, \mathbf{c}, S)$
(15)	(T, \mathbf{c}, C_0)	\rightarrow	T

$$\begin{aligned} S &::= \{D\}^? C_0 \\ D &::= \{\{C_1(\Lambda) : \}^* C_3(\Lambda) : \}^? \{C_2(x) : \}^* \\ &\quad | \{D\}^? C_5(\text{NNF}) : \{C_4(\Lambda) : \}^* \{C_2(x) : \}^* \end{aligned}$$

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If the current continuation is C_1 , the next is always C_1 or C_3 .

If the current continuation is C_3 , the next is never C_1 nor C_3 .

(1)	T	\rightarrow	(T, \mathbf{n}, C_0)
(2)	(x, \mathbf{n}, S)	\rightarrow	(x, \mathbf{c}, S)
(3)	$(\lambda x. B, \mathbf{n}, S)$	\rightarrow	$(B, \mathbf{n}, C_2(x) : S)$
(4)	$(M N, \mathbf{n}, S)$	\rightarrow	$(M, \mathbf{w}, C_3(N) : S)$
(5)	(x, \mathbf{w}, S)	\rightarrow	(x, \mathbf{c}, S)
(6)	$(\lambda x. B, \mathbf{w}, S)$	\rightarrow	$(\lambda x. B, \mathbf{c}, S)$
(7)	$(M N, \mathbf{w}, S)$	\rightarrow	$(M, \mathbf{w}, C_1(N) : S)$
(8)	$(\lambda x. B, \mathbf{c}, C_1(N) : S)$	\rightarrow	if $S \equiv C_{(1 3)}(-) : S'$ then $([N/x]B, \mathbf{w}, S)$ else $([N/x]B, \mathbf{n}, S)$
(9)	$(M \neq \lambda x. B, \mathbf{c}, C_1(N) : S)$	\rightarrow	$(M N, \mathbf{c}, S)$
(10)	$(B, \mathbf{c}, C_2(x) : S)$	\rightarrow	$(\lambda x. B, \mathbf{c}, S)$
(11)	$(\lambda x. B, \mathbf{c}, C_3(N) : S)$	\rightarrow	if $S \equiv C_{(1 3)}(-) : S'$ then $([N/x]B, \mathbf{w}, S)$ else $([N/x]B, \mathbf{n}, S)$
(12)	$(M \neq \lambda x. B, \mathbf{c}, C_3(N) : S)$	\rightarrow	$(M, \mathbf{n}, C_4(N) : S)$
(13)	$(M, \mathbf{c}, C_4(N) : S)$	\rightarrow	$(N, \mathbf{n}, C_5(M) : S)$
(14)	$(N, \mathbf{c}, C_5(M) : S)$	\rightarrow	$(M N, \mathbf{c}, S)$
(15)	(T, \mathbf{c}, C_0)	\rightarrow	T

(1)	T	\rightarrow	(T, \mathbf{n}, C_0)
(2)	(x, \mathbf{n}, S)	\rightarrow	(x, \mathbf{c}, S)
(3)	$(\lambda x. B, \mathbf{n}, S)$	\rightarrow	$(B, \mathbf{n}, C_2(x) : S)$
(4)	$(M N, \mathbf{n}, S)$	\rightarrow	$(M, \mathbf{w}, C_3(N) : S)$
(5)	(x, \mathbf{w}, S)	\rightarrow	(x, \mathbf{c}, S)
(6)	$(\lambda x. B, \mathbf{w}, S)$	\rightarrow	$(\lambda x. B, \mathbf{c}, S)$
(7)	$(M N, \mathbf{w}, S)$	\rightarrow	$(M, \mathbf{w}, C_1(N) : S)$
(8)	$(\lambda x. B, \mathbf{c}, C_1(N) : S)$	\rightarrow	$([N/x]B, \mathbf{w}, S)$
(9)	$(M \neq \lambda x. B, \mathbf{c}, C_1(N) : S)$	\rightarrow	$(M N, \mathbf{c}, S)$
(10)	$(B, \mathbf{c}, C_2(x) : S)$	\rightarrow	$(\lambda x. B, \mathbf{c}, S)$
(11)	$(\lambda x. B, \mathbf{c}, C_3(N) : S)$	\rightarrow	$([N/x]B, \mathbf{n}, S)$
(12)	$(M \neq \lambda x. B, \mathbf{c}, C_3(N) : S)$	\rightarrow	$(M, \mathbf{n}, C_4(N) : S)$
(13)	$(M, \mathbf{c}, C_4(N) : S)$	\rightarrow	$(N, \mathbf{n}, C_5(M) : S)$
(14)	$(N, \mathbf{c}, C_5(M) : S)$	\rightarrow	$(M N, \mathbf{c}, S)$
(15)	(T, \mathbf{c}, C_0)	\rightarrow	T

(1)	T	\rightarrow	(T, \mathbf{n}, C_0)
(2)	(x, \mathbf{n}, S)	\rightarrow	(x, \mathbf{c}, S)
(3)	$(\lambda x. B, \mathbf{n}, S)$	\rightarrow	$(B, \mathbf{n}, C_2(x) : S)$
(4)	$(M N, \mathbf{n}, S)$	\rightarrow	$(M, \mathbf{w}, C_3(N) : S)$
(5)	(x, \mathbf{w}, S)	\rightarrow	(x, \mathbf{c}, S)
(6)	$(\lambda x. B, \mathbf{w}, S)$	\rightarrow	$(\lambda x. B, \mathbf{c}, S)$
(7)	$(M N, \mathbf{w}, S)$	\rightarrow	$(M, \mathbf{w}, C_1(N) : S)$
(8)	$(\lambda x. B, \mathbf{c}, C_1(N) : S)$	\rightarrow	$([N/x]B, \mathbf{w}, S)$
(9)	$(M \neq \lambda x. B, \mathbf{c}, C_1(N) : S)$	\rightarrow	$(M N, \mathbf{c}, S)$
(10)	$(B, \mathbf{c}, C_2(x) : S)$	\rightarrow	$(\lambda x. B, \mathbf{c}, S)$
(11)	$(\lambda x. B, \mathbf{c}, C_3(N) : S)$	\rightarrow	$([N/x]B, \mathbf{n}, S)$
(12)	$(M \neq \lambda x. B, \mathbf{c}, C_3(N) : S)$	\rightarrow	$(M, \mathbf{n}, C_4(N) : S)$
(13)	$(M, \mathbf{c}, C_4(N) : S)$	\rightarrow	$(N, \mathbf{n}, C_5(M) : S)$
(14)	$(N, \mathbf{c}, C_5(M) : S)$	\rightarrow	$(M N, \mathbf{c}, S)$
(15)	(T, \mathbf{c}, C_0)	\rightarrow	T

(1)	T	\rightarrow	(T, \mathbf{n}, C_0)
(2)	(x, \mathbf{n}, S)	\rightarrow	(x, \mathbf{c}, S)
(3)	$(\lambda x. B, \mathbf{n}, S)$	\rightarrow	$(B, \mathbf{n}, C_2(x) : S)$
(4)	$(M N, \mathbf{n}, S)$	\rightarrow	$(M, \mathbf{w}, C_3(N) : S)$
(5)	(x, \mathbf{w}, S)	\rightarrow	(x, \mathbf{c}, S)
(6)	$(\lambda x. B, \mathbf{w}, S)$	\rightarrow	$(\lambda x. B, \mathbf{c}, S)$
(7)	$(M N, \mathbf{w}, S)$	\rightarrow	$(M, \mathbf{w}, C_1(N) : S)$
(8)	$(\lambda x. B, \mathbf{c}, C_1(N) : S)$	\rightarrow	$([N/x]B, \mathbf{w}, S)$
(9)	$(M \neq \lambda x. B, \mathbf{c}, C_1(N) : S)$	\rightarrow	$(M N, \mathbf{c}, S)$
(10)	$(B, \mathbf{c}, C_2(x) : S)$	\rightarrow	$(\lambda x. B, \mathbf{c}, S)$
(11)	$(\lambda x. B, \mathbf{c}, C_3(N) : S)$	\rightarrow	$([N/x]B, \mathbf{n}, S)$
(12)	$(M \neq \lambda x. B, \mathbf{c}, C_3(N) : S)$	\rightarrow	$(M, \mathbf{n}, C_4(N) : S)$
(13)	$(M, \mathbf{c}, C_4(N) : S)$	\rightarrow	$(N, \mathbf{n}, C_5(M) : S)$
(14)	$(N, \mathbf{c}, C_5(M) : S)$	\rightarrow	$(M N, \mathbf{c}, S)$
(15)	(T, \mathbf{c}, C_0)	\rightarrow	T

$$\frac{}{x \Downarrow_{bn} x} \text{ (BN-VAR)} \qquad \frac{}{\lambda x.B \Downarrow_{bn} \lambda x.B} \text{ (BN-ABS)}$$

$$\frac{M \Downarrow_{bn} M' \quad M' \equiv \lambda x.B \quad [N/x]B \Downarrow_{bn} B'}{M N \Downarrow_{bn} B'} \text{ (BN-CON)}$$

$$\frac{M \Downarrow_{bn} M' \quad M' \not\equiv \lambda x.B}{M N \Downarrow_{bn} M' N} \text{ (BN-NEU)}$$

$$\frac{}{x \Downarrow_{no} x} \text{ (NO-VAR)} \qquad \frac{B \Downarrow_{no} B'}{\lambda x.B \Downarrow_{no} \lambda x.B'} \text{ (NO-ABS)}$$

$$\frac{M \Downarrow_{bn} M' \quad M' \equiv \lambda x.B \quad [N/x]B \Downarrow_{no} B'}{M N \Downarrow_{no} B'} \text{ (NO-CON)}$$

$$\frac{M \Downarrow_{bn} M' \quad M' \not\equiv \lambda x.B \quad M' \Downarrow_{no} M'' \quad N \Downarrow_{no} N'}{M N \Downarrow_{no} M'' N'} \text{ (NO-NEU)}$$

$$\frac{}{x \Downarrow_{bn} x} \text{ (BN-VAR)} \qquad \frac{}{\lambda x.B \Downarrow_{bn} \lambda x.B} \text{ (BN-ABS)}$$

$$\frac{M \Downarrow_{bn} M' \quad M' \equiv \lambda x.B \quad [N/x]B \Downarrow_{bn} B'}{M N \Downarrow_{bn} B'} \text{ (BN-CON)}$$

$$\frac{M \Downarrow_{bn} M' \quad M' \not\equiv \lambda x.B}{M N \Downarrow_{bn} M' N} \text{ (BN-NEU)}$$

$$\frac{}{x \Downarrow_{no} x} \text{ (NO-VAR)} \qquad \frac{B \Downarrow_{no} B'}{\lambda x.B \Downarrow_{no} \lambda x.B'} \text{ (NO-ABS)}$$

$$\frac{M \Downarrow_{bn} M' \quad M' \equiv \lambda x.B \quad [N/x]B \Downarrow_{no} B'}{M N \Downarrow_{no} B'} \text{ (NO-CON)}$$

$$\frac{M \Downarrow_{bn} M' \quad M' \not\equiv \lambda x.B \quad M' \Downarrow_{no} M'' \quad N \Downarrow_{no} N'}{M N \Downarrow_{no} M'' N'} \text{ (NO-NEU)}$$

Code in ML (<http://babel.ls.fi.upm.es/~agarcia/papers/PEPM13>)

```
datatype term = IND of int | LAM of term | APP of term * term
datatype whnf = FUN of term | ACC of int * term list
datatype redex = SUB of term * term
datatype found = WHNF of whnf | RED of redex
datatype continuation = C0
```

(\approx 800 LOC)

```
    | C1 of term * continuation
    | C2 of continuation
    | C3 of term * continuation
    | C4 of term * continuation
    | C5 of continuation * whnf
datatype whnf_or_decomposition = WHNF of whnf | DEC of redex * continuation
```

```
(* decompose_cont : continuation * whnf -> whnf_or_decomposition *)
```

```
fun decompose_cont (C0, w) = WHNF w
  | decompose_cont (C1 (n, k), wm) = (case wm of (FUN b) => DEC (SUB (b, n), k)
    | _ => decompose_cont (k, apply_acc (wm, n)))
  | decompose_cont (C2 k, wb) = decompose_cont (k, FUN (embed wb))
  | decompose_cont (C3 (n, k), wm) = (case wm of (FUN b) => DEC (SUB (b, n), k)
    | _ => decompose_nf (embed wm, C4 (n, k)))
  | decompose_cont (C4 (n, k), nm) = decompose_nf (n, C5 (k, nm))
  | decompose_cont (C5 (k, nm), nn) = decompose_cont (k, apply_acc (nm, embed nn))
```

```
(* decompose_whnf : term * continuation -> whnf_or_decomposition *)
```

```
and decompose_whnf (IND n, k) = decompose_cont (k, ACC (n, []))
  | decompose_whnf (LAM b, k) = decompose_cont (k, FUN b)
  | decompose_whnf (APP (m, n), k) = decompose_whnf (m, C1 (n, k))
```

```
(* decompose_nf : term * continuation -> whnf_or_decomposition *)
```

```
and decompose_nf (IND n, k) = decompose_cont (k, ACC (n, []))
  | decompose_nf (LAM b, k) = decompose_nf (b, C2 k)
  | decompose_nf (APP (m, n), k) = decompose_whnf (m, C3 (n, k))
```

```
(* decompose : term -> whnf_or_decomposition *)
```

```
fun decompose t = decompose_nf (t, C0)
```


From normal order to full Krivine machine (KN, 2007)

(21/22)

Red-free normal $\xRightarrow{\text{clos.conv.}}$ Hyb-normal $\xRightarrow{\text{exp.ctrl.}}$ Gen-normal $\xLeftrightarrow{\text{functional}}$ Eval/apply machine $\xRightarrow{\text{inl.simp.}}$ KN machine

Thanks!