Full Reduction in Open Strict Calculus

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IMDEA Software 2011

Motivation

- Typing rules for dependent type systems:

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- Usually by-value (strict) semantics.

Target Strategies

Full-reducing strategies in open strict calculus:

- Full normal forms (NF).

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Values and Irreducible Forms (I)

- Values:

$$\begin{array}{rcl} Val & ::= & x \\ & \mid & \lambda x.B \end{array}$$

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- Weak normal forms in λ_V:

$$
\begin{array}{rcl}
WNF_V & ::= & \lambda x.B \\
 & | & Stuck_W
\end{array}
$$

$$
\begin{array}{rcl}
Stuck_W & ::= & x\, WNF^* \\
 & | & ((\lambda x.B)\, Stuck_W)\, WNF^*
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Note that $Val = (WNF_V - Stuck_W) \cup \{x\}$.

Values and Irreducible Forms (II)

- Normal forms in λ_V:

$$
\begin{array}{rcl}
NF_V & ::= & \lambda x.NF_V \\
 & | & Stuck_V \\
 & & \\
Stuck_V & ::= & x\, {NF_V}^* \\
 & | & ((\lambda x.NF_V)\, Stuck_V)\, {NF_V}^*
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Note that $Val \neq (NF_V - Stuck_V) \cup \{x\}$.

Call-by-value Strategy (cbv)

Uniform standard strategy in λ_V:

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Standardisation Theorem

Theorem

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Weak reduction is not enoguh for implementing joinability!

Full Reduction and Confluence

- Restricting NF's as arguments breaks Church-Rosser:

$$M \equiv (\lambda x.(\lambda y.z)\ (x\ (\lambda x.x\ x)))\ (\lambda x.x\ x)$$

Hybrid Strategies

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$$\frac{M \stackrel{cbn}{\to} M' \equiv \lambda x.B \quad [N/x]B \stackrel{cbn}{\to} S}{M\,N \stackrel{cbn}{\to} S}$$

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Subsidiary

Normal order (nor):

$$\frac{}{x \overset{nor}{\to} x}$$

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$$\frac{\boxed{M \overset{cbn}{\to} M' \equiv \lambda x.B} \qquad [N/x]B \overset{nor}{\to} S}{M\,N \overset{nor}{\to} S}$$

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Hybrid

Rule Template

$$\text{VAR}\ \frac{}{x \ \rightarrow \ x}$$

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Booleans la, $ar1$ and $ar2$ encode the occurrence of the coloured premises in the rule template.

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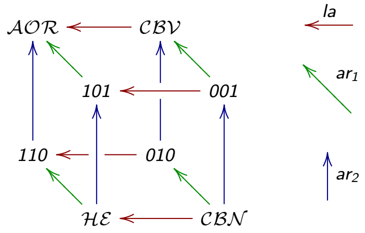
($\sqsubseteq$)



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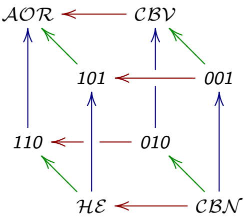

la	0 weakness	1 fullness
ar_1	0 non-strictness	1 strictness
ar_2	0 headness	1 non-headness

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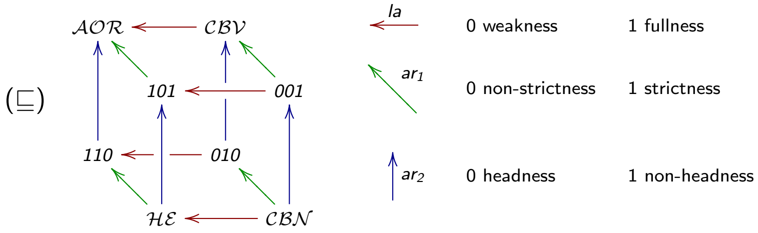

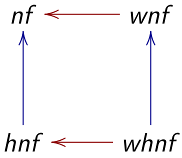

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Hybrid

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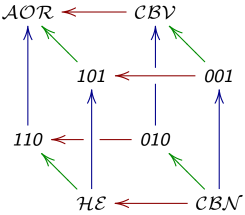
Hybrid

Hybrids in the Non-strict Face of the Cube

$$nor \quad = hyb(\mathit{101}, cbn)$$

($\sqsubseteq$)



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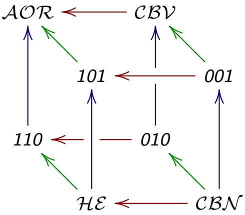

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nor &= hyb(101, cbn) \\
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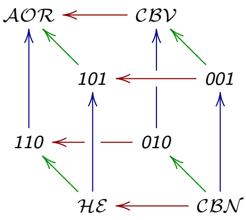

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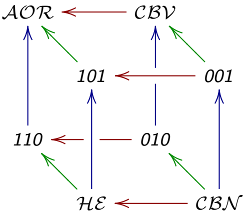


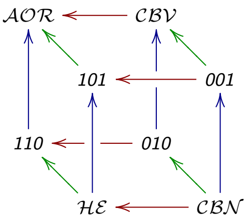
$$\begin{aligned} nor &= hyb(101, cbn) \\ hn &= hyb(101, he) \\ hr &= hyb(he, cbn) \end{aligned}$$

Conjecture

$$nor = hn = hyb(101, 001)$$

Hybrids in the Strict Face of the Cube

($\sqsubseteq$)



$$ha \quad = hyb(aor, cbv)$$

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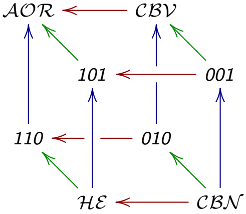

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ha is not standard in λ_V!

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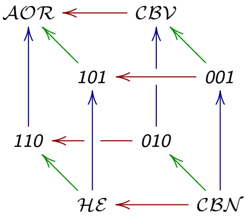

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$M\ N$

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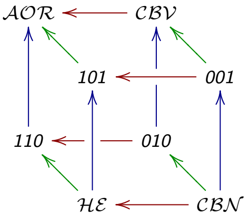

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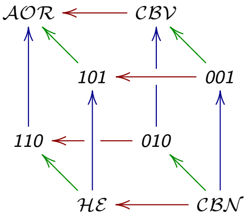

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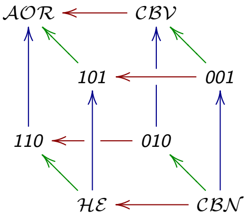

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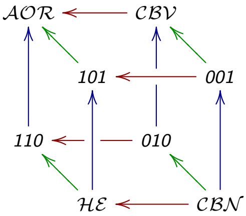

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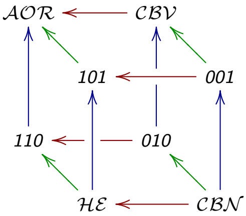

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Standard Hybridisation Operator

$$\frac{}{x \stackrel{sub}{\to} x}$$

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Hybrid

Strict Face with Open terms

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  - Open terms may occur as subterms of well-formed (closed) terms when performing full reduction.

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$$\text{APP}\ \frac{M \ \rightarrow \ M' \not\equiv \lambda x.B \qquad N \stackrel{ar_2}{\rightarrow} N'}{M\,N \ \rightarrow M'\ N'}$$

Cube and Hybridisation for Open Terms

Extending the cube and the standard hybridisation operator with the new rule is trivial.

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- Sestoft 2002: accounts for full reduction, but doesn't account for standardisation.
- Leroy and Gregoire 2002: accounts for full reduction and standardisation, but doesn't account for confluence in λ_V.

Conclusions

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- Beta Cube and hybridisation to articulate strategies space.
- Plotkin's meta-theory leads the improvements in cube and hybridisation.