

Reference Card

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April 2008

- **Logic** Rules of Inference: slides 2 to 6.
- **Equality** Rules of Inference: slide 7.
- **Set-theoretic** Axioms and Definitions: slides 8 to 20.
- **Syntax** of Event-B: slides 21 to 23.
- **Proof Obligation** Rules: slides 24 to 36.
- **ASCII Representations** of the Math. Symbols: slides 37 to 41.

Basic Inference Rules of Mathematical Reasoning

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$$\frac{}{H, P \vdash P} \text{ HYP}$$

$$\frac{H \vdash Q}{H, P \vdash Q} \text{ MON}$$

$$\frac{H \vdash P \quad H, P \vdash Q}{H \vdash Q} \text{ CUT}$$

Propositional Calculus Rules of Inference (1)

3

- Rules about conjunction

$$\frac{H, P, Q \vdash R}{H, P \wedge Q \vdash R} \text{ AND_L}$$

$$\frac{H \vdash P \quad H \vdash Q}{H \vdash P \wedge Q} \text{ AND_R}$$

- Rules about implication

$$\frac{H, P, Q \vdash R}{H, P, P \Rightarrow Q \vdash R} \text{ IMP_L}$$

$$\frac{H, P \vdash Q}{H \vdash P \Rightarrow Q} \text{ IMP_R}$$

Note: Rules with a **double horizontal line** can be applied in **both directions**

- Rules about negation

$$\frac{}{P, \neg P \vdash Q} \text{ NOT_L} \quad \frac{}{\perp \vdash P} \text{ CNTR}$$

$$\frac{H, P \vdash Q \quad H, P \vdash \neg Q}{H \vdash \neg P} \text{ NOT_R}$$

$$\frac{H, \neg P \vdash Q \quad H, \neg P \vdash \neg Q}{H \vdash P} \text{ NOT_R}$$

- Rules about disjunction

$$\frac{H, P \vdash R \quad H, Q \vdash R}{H, P \vee Q \vdash R} \text{ OR_L}$$

$$\frac{H \vdash P}{H \vdash P \vee Q} \text{ OR_R1}$$

$$\frac{H \vdash Q}{H \vdash P \vee Q} \text{ OR_R2}$$

- Transforming a disjunctive goal

$$\frac{H, \neg P \vdash Q}{H \vdash P \vee Q} \text{ NEG}$$

Predicate Calculus Rules of Inference

6

$$\frac{H, \forall x \cdot P(x), P(E) \vdash Q}{H, \forall x \cdot P(x) \vdash Q} \text{ ALL_L} \quad \frac{H \vdash P(x)}{H \vdash \forall x \cdot P(x)} \text{ ALL_R}$$

$$\frac{H, P(x) \vdash Q}{H, \exists x \cdot P(x) \vdash Q} \text{ XST_L} \quad \frac{H \vdash P(E)}{H \vdash \exists x \cdot P(x)} \text{ XST_R}$$

- In rule **ALL_L** and **XST_R**, **E** is an expression

- In rule **ALL_R** and **XST_L**, variable **x** is not free in **H**.

Equality Rules of Inference

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$$\frac{H(F), E = F \vdash P(F)}{H(E), E = F \vdash P(E)} \text{ EQ_LR} \quad \frac{H(E), E = F \vdash P(E)}{H(F), E = F \vdash P(F)} \text{ EQ_RL}$$

$$\frac{}{\vdash E = E} \text{ EQL}$$

$$\frac{H \vdash E = G \wedge F = I}{H \vdash E \mapsto F = G \mapsto I} \text{ PAIR}$$

These axioms are defined by **equivalences**.

Left Part	Right Part
$E \mapsto F \in S \times T$	$E \in S \wedge F \in T$
$S \in \mathbb{P}(T)$	$\forall x \cdot x \in S \Rightarrow x \in T$
$E \in \{x \cdot x \in S \wedge P(x) \mid F(x)\}$	$\exists x \cdot x \in S \wedge P(x) \wedge E = F(x)$
$E \in \{x \mid x \in S \wedge P(x)\}$	$E \in S \wedge P(E)$

Left Part	Right Part
$S \subseteq T$	$S \in \mathbb{P}(T)$
$S = T$	$S \subseteq T \wedge T \subseteq S$

The first rule is just a **syntactic extension**

The second rule is the **Extensionality Axiom**

$E \in S \cup T$	$E \in S \vee E \in T$
$E \in S \cap T$	$E \in S \wedge E \in T$
$E \in S \setminus T$	$E \in S \wedge E \notin T$
$E \in \{a, \dots, b\}$	$E = a \vee \dots \vee E = b$
$E \in \emptyset$	\perp

$E \in \text{union}(S)$	$\exists s \cdot s \in S \wedge E \in s$
$E \in \bigcup x \cdot x \in S \wedge P(x) \mid T(x)$	$\exists x \cdot x \in S \wedge P(x) \wedge E \in T(x)$
$E \in \text{inter}(S)$	$\forall s \cdot s \in S \Rightarrow E \in s$
$E \in \bigcap x \cdot x \in S \wedge P(x) \mid T(x)$	$\forall x \cdot x \in S \wedge P(x) \Rightarrow E \in T(x)$

Well-definedness condition for case 3: $S \neq \emptyset$

Well-definedness condition for case 4: $\exists x \cdot x \in S \wedge P(x)$

Left Part	Right Part
$r \in S \leftrightarrow T$	$r \subseteq S \times T$
$E \in \text{dom}(r)$	$\exists y \cdot E \mapsto y \in r$
$F \in \text{ran}(r)$	$\exists x \cdot x \mapsto F \in r$
$E \mapsto F \in r^{-1}$	$F \mapsto E \in r$

Left Part	Right Part
$r \in S \leftrightarrow T$	$r \in S \leftrightarrow T \wedge \text{ran}(r) = T$
$r \in S \leftrightarrow T$	$r \in S \leftrightarrow T \wedge \text{dom}(r) = T$
$r \in S \leftrightarrow T$	$r \in S \leftrightarrow T \wedge r \in S \leftrightarrow T$

Left Part	Right Part
$E \mapsto F \in S \triangleleft r$	$E \in S \wedge E \mapsto F \in r$
$E \mapsto F \in r \triangleright T$	$E \mapsto F \in r \wedge F \in T$
$E \mapsto F \in S \triangleleft r$	$E \notin S \wedge E \mapsto F \in r$
$E \mapsto F \in r \triangleright T$	$E \mapsto F \in r \wedge F \notin T$

$F \in r[w]$	$\exists x \cdot x \in w \wedge x \mapsto F \in r$
$E \mapsto F \in (p ; q)$	$\exists x \cdot E \mapsto x \in p \wedge x \mapsto F \in q$
$p \triangleleft q$	$(\text{dom}(q) \triangleleft p) \cup q$
$E \mapsto F \in \text{id}(S)$	$E \in S \wedge F = E$

$E \mapsto (F \mapsto G) \in p \otimes q$	$E \mapsto F \in p \wedge E \mapsto G \in q$
$(E \mapsto F) \mapsto G \in \text{prj}_1(S, T)$	$E \in S \wedge F \in T \wedge G = E$
$(E \mapsto F) \mapsto G \in \text{prj}_2(S, T)$	$E \in S \wedge F \in T \wedge G = F$
$(E \mapsto G) \mapsto (F \mapsto H) \in p \parallel q$	$E \mapsto F \in p \wedge G \mapsto H \in q$

Given a relation r such that $r \in S \leftrightarrow S$

$r = r^{-1}$	r is symmetric
$r \cap r^{-1} = \emptyset$	r is asymmetric
$r \cap r^{-1} \subseteq \text{id}(S)$	r is antisymmetric
$\text{id}(S) \subseteq r$	r is reflexive
$r \cap \text{id}(S) = \emptyset$	r is irreflexive
$r; r \subseteq r$	r is transitive

Left Part	Right Part
$f \in S \leftrightarrow T$	$f \in S \leftrightarrow T \wedge (f^{-1}; f) = \text{id}(\text{ran}(f))$
$f \in S \rightarrow T$	$f \in S \leftrightarrow T \wedge S = \text{dom}(f)$
$f \in S \rightsquigarrow T$	$f \in S \leftrightarrow T \wedge f^{-1} \in T \leftrightarrow S$
$f \in S \rightharpoonup T$	$f \in S \rightarrow T \wedge f^{-1} \in T \leftrightarrow S$

Left Part	Right Part
$f \in S \twoheadrightarrow T$	$f \in S \rightarrow T \wedge T = \text{ran}(f)$
$f \in S \twoheadrightarrow T$	$f \in S \rightarrow T \wedge T = \text{ran}(f)$
$f \in S \twoheadrightarrow T$	$f \in S \twoheadrightarrow T \wedge f \in S \rightarrow T$

Given a **partial function** f , we have

Left Part	Right Part
$F = f(E)$	$E \mapsto F \in f$

Well-definedness conditions: **f is a partial function**

Machine Structure

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```

machine
  < machine_identifier >
refines *
  < machine_identifier >
sees *
  < context_identifier >
  ...
variables
  < variable_identifier >
  ...
invariants
  < label >: < predicate >
  ...
theorems *
  < label >: < predicate >
  ...
events
  initialisation ...
  ...
variant *
  < variant >
end

```

- Each machine has exactly one **initialisation event**
- All keyword sections are predefined in the Rodin Platform
- All labels are generated automatically by the Rodin Platform (but can be modified)

```

context
  < context_identifier >
extends *
  < context_identifier >
  ...
sets *
  < set_identifier >
  ...
constants *
  < constant_identifier >
  ...
axioms *
  < label >: < predicate >
  ...
theorems *
  < label >: < predicate >
  ...
end

```

- Sections with "*" might be empty
- All keyword sections are predefined in the Rodin Platform
- All labels are generated automatically by the Rodin Platform (but can be modified)

Event Structure

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```

< event_identifier > ≐
status
  { ordinary, convergent, anticipated }
refines *
  < event_identifier >
  ...
any *
  < parameter_identifier >
  ...
where *
  < label >: < predicate >
  ...
with *
  < label >: < witness >
  ...
then *
  < label >: < action >
  ...
end

```

- Notice that keyword "where" becomes "when" in the Rodin Platform Pretty Print when there is no "any".
- Again, all keyword sections are predefined in the Rodin Platform.
- All labels are generated automatically by the Rodin Platform (but can be modified)

<pre> evt any x where G(x, s, c, v) then v : BAP(x, s, c, v, v') end </pre>	<p><i>s</i> : seen sets <i>c</i> : seen constants <i>v</i> : variables <i>A(s, c)</i> : seen axioms and thms <i>I(s, c, v)</i> : invariants and thms. <i>evt</i> : specific event <i>x</i> : event parameters <i>G(x, s, c, v)</i> : event guards <i>BAP(x, s, c, v, v')</i> : event before-after predicate <i>inv(s, c, v')</i> : modified specific invariant</p>
--	---

Axioms Invariants Guards of the event Before-after predicate of the event \vdash Modified Specific Invariant	<i>evt/inv/INV</i>
---	--------------------

$A(s, c)$
 $I(s, c, v)$
 $G(x, s, c, v)$
 $BAP(x, s, c, v, v')$
 \vdash
 $inv(s, c, v')$

<pre> evt any x where G(x, s, c, v) then v : BAP(x, s, c, v, v') end </pre>	<p><i>s</i> : seen sets <i>c</i> : seen constants <i>v</i> : variables <i>A(s, c)</i> : seen axioms and thms <i>I(s, c, v)</i> : invariants and thms. <i>evt</i> : specific event <i>x</i> : event parameters <i>G(x, s, c, v)</i> : event guards <i>BAP(x, s, c, v, v')</i> : event action</p>
--	---

Axioms Invariants Guards of the event \vdash $\exists v' \cdot$ Before-after predicate	<i>evt/act/FIS</i>
--	--------------------

$A(s, c)$
 $I(s, c, v)$
 $G(x, s, c, v)$
 \vdash
 $\exists v' \cdot BAP(x, s, c, v, v')$

- In case of the initialization event, $I(s, c, v)$ is removed from the hypotheses

<pre> evt0 any x where g(x, s, c, v) ... then ... end </pre>	<pre> evt refines evt0 any y where H(y, s, c, w) with x : W(x, y, s, c, w) then ... end </pre>	<p><i>s</i> : seen sets <i>c</i> : seen constants <i>v</i> : abstract variables <i>w</i> : concrete variables <i>A(s, c)</i> : seen axioms and thms <i>I(s, c, v)</i> : abs. invariants and thms. <i>J(s, c, v, w)</i> : conc. invariants and thms. <i>evt</i> : specific concrete event <i>x</i> : abstract event parameter <i>y</i> : concrete event parameter <i>g(x, s, c, v)</i> : abstract event specific guard <i>H(y, s, c, w)</i> : concrete event guards</p>
--	--	---

Axioms Abstract invariants and thms. Concrete invariants and thms. Concrete event guards witness predicate \vdash Abstract event specific guard	<i>evt/grd/GRD</i>
---	--------------------

$A(s, c)$
 $I(s, c, v)$
 $J(s, c, v, w)$
 $H(y, s, c, w)$
 $W(x, y, s, c, w)$
 \vdash
 $g(x, s, c, v)$

<pre> evt0 any x where ... then v : BA1(v, v', ...) end </pre>	<pre> evt refines evt0 any y where H(y, s, c, w) with x : W1(x, y, s, c, w) v' : W2(y, v', s, c, w) then w : BA2(w, w', ...) end </pre>	<p><i>s</i> : seen sets <i>c</i> : seen constants <i>v</i> : abstract vrbles <i>w</i> : concrete vrbles <i>A(s, c)</i> : seen axioms and thms <i>I(s, c, v)</i> : abs. invariants and thms. <i>J(s, c, v, w)</i> : conc. invariants and thms. <i>evt</i> : concrete event <i>x</i> : abstract prm <i>y</i> : concrete prm <i>H(y, s, c, w)</i> : concrete guards <i>BA1(v, v')</i> : abstract action <i>BA2(w, w')</i> : concrete action</p>
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Axioms Abstract invariants and thms. Concrete invariants and thms. Concrete event guards witness predicate Concrete before-after predicate \vdash Abstract before-after predicate	<i>evt/act/SIM</i>
--	--------------------

$A(s, c)$
 $I(s, c, v)$
 $J(s, c, v, w)$
 $H(y, s, c, w)$
 $W1(x, y, s, c, w)$
 $W2(y, v', s, c, w)$
 $BA2(w, w', ...)$
 \vdash
 $BA1(v, v', ...)$

- It is simplified when there are no parameters

```

machine
  m
refines
...
sees
...
variables
  v
invariants and thms.
  I(s, c, v)
theorems
...
events
...
variant
  n(s, c, v)
end
    
```

```

evt
status
  convergent
  any x where
    G(x, s, c, v)
  then
    A
  end
    
```

s : seen sets
c : seen constants
v : variables
A(s, c) : seen axioms and thms
I(s, c, v) : abs. invariants and thms.
J(s, c, v, w) : conc. invariants and thms.
evt : specific event
x : event parameters
G(x, s, c, v) : event guards
n(s, c, v) : numeric variant

Axioms Abstract invariants and thms. Concrete invariants and thms. Event guards ⊢ a numeric variant is a natural number	<i>evt</i> /NAT
---	-----------------

A(s, c)
I(s, c, v)
J(s, c, v, w)
G(x, s, c, v)
 ⊢ $n(s, c, v) \in \mathbb{N}$

```

machine
  m
refines
...
sees
...
variables
  v
invariants and thms.
  J(s, c, v, w)
theorems
...
events
...
variant
  t(s, c, v)
end
    
```

s : seen sets
c : seen constants
v : variables
A(s, c) : seen axioms and thms
I(s, c, v) : abs. invariants and thms.
J(s, c, v, w) : conc. invariants and thms.
t(s, c, v) : set variant

Axioms Abstract invariants and thms. Concrete invariants and thms. ⊢ Finiteness of set variant	FIN
---	-----

A(s, c)
I(s, c, v)
J(s, c, v, w)
 ⊢ $\text{finite}(t(s, c, v))$

```

evt
status
  convergent
  any x where
    G(x, s, c, w)
  then
    v :| BAP(x, s, c, w, w')
  end
    
```

s : seen sets
c : seen constants
v : variables
A(s, c) : seen axioms and thms
I(s, c, v) : abs. invariants and thms.
J(s, c, v, w) : conc. invariants and thms.
evt : specific event
x : event parameters
G(x, s, c, v) : event guards
BAP(x, s, c, w, w') : event before-after predicate
n(s, c, w) : numeric variant

Axioms Abstract invariants and thms. Concrete invariants and thms. Guards of the event Before-after predicate of the event ⊢ Modified variant smaller than variant	<i>evt</i> /VAR
---	-----------------

A(s, c)
I(s, c, v)
J(s, c, v, w)
G(x, s, c, w)
BAP(x, s, c, w, w')
 ⊢ $n(s, c, w') < n(s, c, w)$

```

evt
status
  convergent
  any x where
    G(x, s, c, w)
  then
    v :| BAP(x, s, c, w, w')
  end
    
```

s : seen sets
c : seen constants
v : variables
A(s, c) : seen axioms and thms
I(s, c, v) : abs. invariants and thms.
J(s, c, v, w) : conc. invariants and thms.
evt : specific event
x : event parameters
G(x, s, c, v) : event guards
BAP(x, s, c, w, w') : event before-after predicate
t(s, c, w) : set variant

Axioms Abstract Invariants Concrete Invariants Guards of the event Before-after predicate of the event ⊢ Modified variant strictly included in variant	<i>evt</i> /VAR
---	-----------------

A(s, c)
I(s, c, v)
J(s, c, v, w)
G(x, s, c, v)
BAP(x, s, c, w, w')
 ⊢ $t(s, c, w') \subset t(s, c, w)$


```

evt
refines
  evt0
any
  y
where
   $H(y, s, c, w)$ 
with
   $x : W(x, y, s, c, w)$ 
then
  ...
end
    
```

s : seen sets
 c : seen constants
 v : abstract variables
 w : concrete variables
 $A(s, c)$: seen axioms and thms
 $I(s, c, v)$: abs. invts. and thms.
 $J(s, c, v, w)$: conc. invts. and thms.
 evt : specific concrete event
 x : abstract event parameter
 y : concrete event parameter
 $H(y, s, c, w)$: concrete event guards
 $W(x, y, s, c, w)$: witness predicate

```

context
  ctx
extends
  ...
sets
   $s$ 
constants
   $c$ 
axioms
   $A(s, c)$ 
theorems
  ...
   $thm : P(s, c)$ 
  ...
end
    
```

s : seen sets
 c : seen constants
 $A(s, c)$: seen axioms and previous thms
 $P(s, c)$: specific theorem

Axioms Abstract invariants and thms. Concrete invariants and thms. Concrete event guards \vdash $\exists x \cdot \text{Witness}$	$evt/x/WFIS$
---	--------------

$A(s, c)$
 $I(s, c, v)$
 $J(s, c, v, w)$
 $H(y, s, c, w)$
 \vdash
 $\exists x \cdot W(x, y, s, c, w)$

Axioms \vdash Theorem	thm/THM
-------------------------------	-----------

$A(s, c)$
 \vdash
 $P(s, c)$

```

machine
  m0
refines
  ...
sees
  ...
variables
   $v$ 
invariants and thms.
   $I(s, c, v)$ 
theorems
  ...
   $thm : P(s, c, v)$ 
  ...
events
  ...
end
    
```

s : seen sets
 c : seen constants
 v : variables
 $A(s, c)$: seen axioms and thms
 $I(s, c, v)$: invariants and previous thms.
 $P(s, c, v)$: specific theorem

Axioms Invariants \vdash Theorem	thm/THM
---	-----------

$A(s, c)$
 $I(s, c, v)$
 \vdash
 $P(s, c, v)$

- It depends on the potentially ill-defined expression

inter(S)	$S \neq \emptyset$
$\bigcap x \cdot x \in S \wedge P(x) \mid T(x)$	$\exists x \cdot x \in S \wedge P(x)$
$f(E)$	f is a partial function $E \in \text{dom}(f)$
E/F	$F \neq 0$
$E \text{ mod } F$	$F \neq 0$
card(S)	finite(S)
min(S)	$S \subseteq \mathbb{Z}$ $\exists x \cdot x \in \mathbb{Z} \wedge (\forall n \cdot n \in S \Rightarrow x \leq n)$
max(S)	$S \subseteq \mathbb{Z}$ $\exists x \cdot x \in \mathbb{Z} \wedge (\forall n \cdot n \in S \Rightarrow x \geq n)$

evt01
any
<i>x</i>
where
<i>G1(x, s, c, v)</i>
then
<i>A</i>
end

evt02
any
<i>x</i>
where
<i>G2(x, s, c, v)</i>
then
<i>A</i>
end

evt
refines
evt01
evt02
any
<i>x</i>
where
<i>H(x, s, c, v)</i>
then
<i>A</i>
end

s : seen sets
c : seen constants
v : abstract vrbles
A(s, c) : seen axioms and thms
I(s, c, v) : abs. invts. and thms.
evt : concrete event
x : similar prm
H(x, s, c, v) : concrete guards
G1(x, s, c, v) : abstract event guards
G2(x, s, c, v) : abstract event guards
A : similar abs. and cnc. actions

Axioms
Abstract invariants and thms.
Concrete event guards
⊢
Disjunction of abstract guards

evt/MRG

A(s, c)
I(s, c, v)
H(x, s, c, v)
 ⊢
G1(x, s, c, v) ∨ G2(x, s, c, v)

- Atomic Symbols

ASCII	Symbol
true	⊤
false	⊥
INT	ℤ

ASCII	Symbol
NAT	ℕ
NAT1	ℕ ₁
BOOL	BOOL

ASCII	Symbol
TRUE	TRUE
FALSE	FALSE
{}	∅

- Assignment Operators

ASCII	Symbol
:=	:=

ASCII	Symbol
:	:

ASCII	Symbol
::	:∈

- Unary Operators

ASCII	Symbol
not	¬
finite	finite
card	card
POW	ℙ
POW1	ℙ ₁

ASCII	Symbol
union	union
inter	inter
dom	dom
ran	ran
prj1	prj ₁

ASCII	Symbol
prj2	prj ₂
id	id
min	min
max	max
-	-

- Binary Operators

ASCII	Symbol
&	∧
or	∨
=>	⇒
<=>	⇔
=	=
/=	≠
:	∈
<<:	⊂

ASCII	Symbol
/<<:	⊄
<:	⊆
/<:	⊈
<	<
<=	≤
>	>
>=	≥
/:	∉

ASCII	Symbol
-> or ,,	↦
<->	↔
<<->	↔↔
<->>	↔↔
<<->>	↔↔↔
+->	→
-->	→
+-->	→

- Binary Operators (Cont.)

ASCII	Symbol
-->>	\rightarrow
>+>	\mapsto
>->	\rightsquigarrow
>->>	\rightsquigarrow
/\	\subset
\	\supset
\	\setminus
**	\times

ASCII	Symbol
<+	\Leftarrow
	\parallel
>>	\otimes
;	$;$
<	\triangleleft
<<	\triangleleft
>	\triangleright
>>	\triangleright

ASCII	Symbol
*	$*$
/	\div
mod	mod
..	\ddots
^	\wedge
~	\sim
+	$+$
-	$-$

- Quantifiers

ASCII	Symbol
!	\forall
#	\exists
%	λ

ASCII	Symbol
UNION	\cup
INTER	\cap

ASCII	Symbol
	$ $
.	\cdot

- Bracketing

ASCII	Symbol
($($
)	$)$

ASCII	Symbol
[$[$
]	$]$

ASCII	Symbol
{	$\{$
}	$\}$